

# LIIOUVILLE HYPERSURFACES AND CONNECT SUM COBORDISMS

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**ABSTRACT.** The purpose of this paper is to introduce *Liouville hypersurfaces* in contact manifolds, which generalize ribbons of Legendrian graphs and pages of supporting open books. Liouville hypersurfaces are used to define a gluing operation for contact manifolds called the *Liouville connect sum*. Performing this operation on a contact manifold  $(M, \xi)$  gives an exact – and in many cases, Stein – cobordism from  $(M, \xi)$  to the surgered manifold. These cobordisms are used to establish the existence of “fillability” and “non-vanishing contact homology” monoids in symplectomorphism groups of Liouville domains, study the symplectic fillability of a family of contact manifolds which fiber over the circle, associate cobordisms to certain branched coverings of contact manifolds, and construct exact symplectic cobordisms that do not admit Stein structures. The Liouville connect sum generalizes the Weinstein handle attachment and is used to extend the definition of contact  $(1/k)$ -surgery along Legendrian knots in contact 3-manifolds to contact  $(1/k)$ -surgery along Legendrian spheres in contact manifolds of arbitrary dimension. This is used to construct exotic contact structures on 5- and 13-dimensional spheres after establishing that  $S^2$  and  $S^6$  are the only spheres along which generalized Dehn twists smoothly square to the identity mapping. The exoticty of these contact structures implies that Dehn twists along  $S^2$  and  $S^6$  do not symplectically square to the identity.

## 1. INTRODUCTION

**1.1. Preliminaries.** A *contact manifold* is a pair  $(M, \xi)$  where  $M$  is an oriented  $(2n + 1)$ -dimensional manifold and  $\xi$  is a globally cooriented  $(2n)$ -plane field on  $M$  such that there is a 1-form  $\alpha \in \Omega^1(M)$  satisfying

$$\text{Ker}(\alpha) = \xi \quad \text{and} \quad \alpha \wedge (d\alpha)^n > 0$$

with respect to the orientation on  $M$ . We also say that  $\xi$  is a *contact structure* on  $M$ . A 1-form  $\alpha$  satisfying the above equation is a *contact form* for  $(M, \xi)$ .

An oriented, codimension-1 submanifold  $M$  of a symplectic manifold  $(W, \omega)$  is a *contact hypersurface* [Wei79] if there is a neighborhood  $N(M)$  of  $M$  such that  $\omega = d\lambda$  for some  $\lambda \in \Omega^1(N(M))$  and the vector field  $X$  determined by  $\omega(X, *) = \lambda$  is positively transverse to  $M$ . This implies that  $\lambda|_{TM}$  is a contact form on  $M$ . In this paper, we will be primarily concerned with the following class of symplectic manifolds whose boundaries are contact hypersurfaces:

**Definition 1.1.** A Liouville domain is a pair  $(\Sigma, \beta)$  where

- (1)  $\Sigma$  is a smooth, compact manifold with boundary,
- (2)  $\beta \in \Omega^1(\Sigma)$  is such that  $d\beta$  is a symplectic form on  $\Sigma$ , and
- (3) the unique vector field  $X_\beta$  satisfying  $d\beta(X_\beta, *) = \beta$  points out of  $\partial\Sigma$  transversely.

The vector field  $X_\beta$  on  $\Sigma$  described above is called the Liouville vector field for  $(\Sigma, \beta)$ .

We say that two Liouville 1-forms  $\beta$  and  $\beta'$  are *homotopic* if there is a smooth family  $\beta_t$ ,  $t \in [0, 1]$ , of Liouville 1-forms on  $\Sigma$  with  $\beta_0 = \beta$  and  $\beta_1 = \beta'$ . In such a situation, the contact structures  $\text{Ker}(\beta)$  and  $\text{Ker}(\beta')$  on  $\partial\Sigma$  are isotopic by Gray’s stability theorem [MS99, §3.4].

**Example 1.2.** Denote by  $\mathbb{D}^{2n+2}$  the unit disk in  $\mathbb{R}^{2n+2}$ . The *standard 1-form* on  $\mathbb{D}^{2n+2}$  is

$$\lambda_{std} = \frac{1}{2} \sum_{j=1}^{n+1} (x_j dy_j - y_j dx_j)$$

in terms of coordinates  $(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1})$ . The *standard contact sphere*, denoted  $(S^{2n+1}, \xi_{std})$ , is the boundary of  $\mathbb{D}^{2n+2}$  with  $\xi_{std} = \text{Ker}(\lambda_{std}|_{TS^3})$ .

**1.2. Liouville submanifolds of contact manifolds.** If  $M$  is a contact hypersurface in a symplectic manifold  $(W, \omega)$ , then, on a sufficiently small neighborhood  $N(M)$  of  $M$ ,

- (1) there is a vector field  $X$  transverse to  $M$  which is a *symplectic dilation*, i.e., satisfies  $\mathcal{L}_X \omega = \omega$ , and
- (2) the contact structure on  $M$  determines the conformal class of  $\omega$  on  $N(M)$ .

We would like to define a class of codimension-1 submanifolds of contact manifolds with analogous properties. One natural candidate definition would be that of a *convex hypersurface* introduced in [Gi91, §1.3] and reviewed in Section 3.4. The purpose of this paper is to study a more restricted class of hypersurfaces in contact manifolds and some related constructions.

**Definition 1.3.** Let  $(M, \xi)$  be a  $(2n+1)$ -dimensional contact manifold and let  $(\Sigma, \beta)$  be a  $2k$ -dimensional Liouville domain. A Liouville embedding  $i : (\Sigma, \beta) \rightarrow (M, \xi)$  is an embedding  $i : \Sigma \rightarrow M$  such that there exists a contact form  $\alpha$  for  $(M, \xi)$  for which  $i^* \alpha = \beta$ . The image of a Liouville embedding will be called a Liouville submanifold and will be denoted by  $(\Sigma, \beta) \subset (M, \xi)$ . When  $k = n$ , we say that  $(\Sigma, \beta) \subset (M, \xi)$  is a Liouville hypersurface in  $(M, \xi)$ .

**Remark 1.4.** In Section 4 it is shown that every Liouville submanifold in a contact manifold  $(M, \xi)$  can be realized as the zero section of a symplectic disk bundle whose total space is a Liouville hypersurface in  $(M, \xi)$ .

Definition 1.3 implies that the boundary  $\partial \Sigma$  of a Liouville hypersurface  $(\Sigma, \beta) \subset (M, \xi)$  is a codimension 2 contact submanifold of  $(M, \xi)$  when oriented as the boundary of  $\Sigma$ . For example, when  $(M, \xi)$  is 3-dimensional, the boundary of a Liouville hypersurface is a (positive) transverse link. Loosely speaking, Liouville hypersurfaces in a contact manifold  $(M, \xi)$  are positive regions of convex hypersurfaces in  $(M, \xi)$ . This will be made more precise in Proposition 6.3.

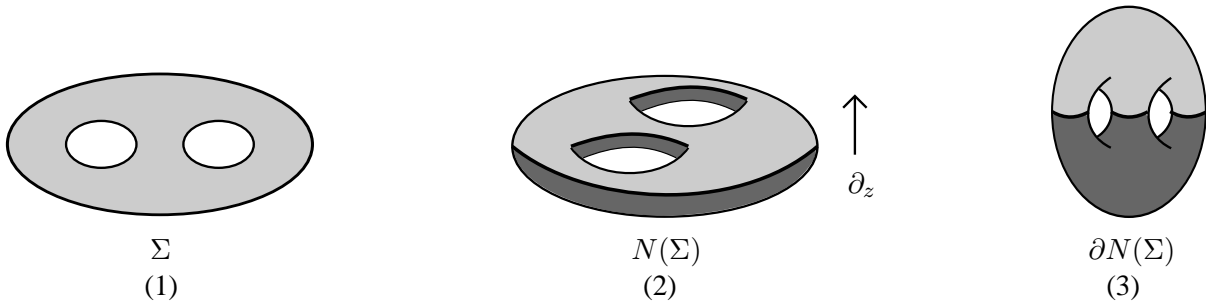


FIGURE 1. Moving from left to right we have (1) a Liouville hypersurface  $(\Sigma, \beta)$  represented by a pair of pants, (2) a neighborhood  $\mathcal{N}(\Sigma)$  of  $\Sigma$  represented by a handlebody, and (3)  $\partial \mathcal{N}(\Sigma)$  depicted as an abstract surface. In schematic figures Liouville domains and hypersurfaces will be represented by pairs-of-pants, unless otherwise stated. Whenever we draw convex hypersurfaces, we lightly shade the positive regions and heavily shade the negative regions. See Section 3.4 for further explanation.

Every Liouville hypersurface  $(\Sigma, \beta) \subset (M, \xi)$  admits a neighborhood of the form

$$N(\Sigma) = [-\epsilon, \epsilon] \times \Sigma \quad \text{on which} \quad \alpha = dz + \beta$$

where  $z$  is a coordinate on  $[-\epsilon, \epsilon]$ . After rounding the edges  $(\partial[-\epsilon, \epsilon]) \times \partial\Sigma$  of  $[-\epsilon, \epsilon] \times \Sigma$ , we obtain a neighborhood  $\mathcal{N}(\Sigma)$  of  $\Sigma$  for which  $\partial\mathcal{N}(\Sigma)$  is a smooth convex surface in  $(M, \xi)$  with contact vector field  $z\partial_z + X_\beta$  and dividing set  $\{0\} \times \partial\Sigma$ ; see Figure 1. More details appear in Section 3.

**Example 1.5.** If  $(\Sigma, \beta) = (\mathbb{D}^{2n}, \lambda_{std}) \subset (M, \xi)$  is a Liouville submanifold of a  $(2n+1)$ -dimensional contact manifold, then the interior of  $\mathcal{N}(\Sigma)$  is a Darboux ball. If  $L \subset (M, \xi)$  is an isotropic sphere with trivial normal bundle and  $\alpha$  is a contact form for  $(M, \xi)$ , then we can find a compact hypersurface  $\Sigma$  with non-empty boundary in  $(M, \xi)$  which deformation retracts onto  $L$  and is diffeomorphic to a tubular neighborhood of the zero section of the bundle  $\mathbb{R}^{2m} \oplus T^*L \rightarrow L$  for which  $\alpha|_{T\Sigma} = \lambda_{std} - \lambda_{can}$ . Here  $m + \dim(L) = n$  and  $\lambda_{can}$  is the canonical 1-form on  $T^*L$  described in Example 2.1, below.

Similar statements hold without the assumptions that the normal bundle of  $L$  is trivial or that  $L$  is a sphere, by the results in Section 4. The case  $(\Sigma, \beta) = (\mathbb{D}^{2n}, \lambda_{std})$  corresponds to the case where  $L$  is a single point. See Examples 2.1 and 1.2 for further explanation.

**1.3. The Liouville connect sum and associated cobordisms.** Convex hypersurfaces provide a simple method of constructing contact manifolds by cut-and-paste. However, examples are hard to find in high ( $> 3$ ) dimensional contact manifolds and it is notoriously difficult to determine how geometric properties of contact structures – such as symplectic fillability, or tightness in dimension three – behave under convex gluing. See, for example, [Hon02].

Using Liouville hypersurfaces, we introduce a special type of convex gluing for contact manifolds called the *Liouville connect sum*. This gluing operation determines an exact symplectic cobordism whose negative boundary is  $(M, \xi)$  and whose positive boundary is the surgered manifold  $\#_{(\Sigma, \beta)}(M, \xi)$ , allowing us to relate symplectic filling properties of  $\#_{(\Sigma, \beta)}(M, \xi)$  to those of  $(M, \xi)$ .

**1.4. Outline of the main construction.** In this section we define the Liouville connect sum and state Theorem 1.9 from which most of our other results will be derived.

Fix a  $2n$ -dimensional Liouville domain  $(\Sigma, \beta)$  and a (possibly disconnected)  $(2n+1)$ -dimensional contact manifold  $(M, \xi)$ . Let  $i_1$  and  $i_2$  be Liouville embeddings of  $(\Sigma, \beta)$  into  $(M, \xi)$  whose images, which we will denote by  $\Sigma_1$  and  $\Sigma_2$ , are disjoint. Let  $\alpha$  be a contact form for  $(M, \xi)$  satisfying  $\alpha|_{T\Sigma_1} = \alpha|_{T\Sigma_2} = \beta$ .

Consider neighborhoods  $\mathcal{N}(\Sigma_1), \mathcal{N}(\Sigma_2) \subset M$  as described in Section 1.2. Taking coordinates  $(z, x)$  on each such neighborhood, where  $x \in \Sigma$  we may consider the mapping

$$\Upsilon : \partial\mathcal{N}(\Sigma_1) \rightarrow \partial\mathcal{N}(\Sigma_2), \quad \Upsilon(z, x) = (-z, x).$$

The map  $\Upsilon$  sends

- (1) the positive region of  $\partial\mathcal{N}(\Sigma_2)$  to the negative region of  $\partial\mathcal{N}(\Sigma_1)$ ,
- (2) the negative region of  $\partial\mathcal{N}(\Sigma_1)$  to the positive region of  $\partial\mathcal{N}(\Sigma_2)$ , and
- (3) the dividing set of  $\partial\mathcal{N}(\Sigma_1)$  to the dividing set of  $\partial\mathcal{N}(\Sigma_2)$

in such a way that we may perform a *convex gluing*. In other words, the map  $\Upsilon$  naturally determines a contact structure  $\#_{((\Sigma, \beta), (i_1, i_2))}\xi$  on the manifold

$$\#_{(\Sigma, (i_1, i_2))}M := \left( M \setminus (N(\Sigma_1) \cup N(\Sigma_2)) \right) / \sim$$

where  $p \sim \Upsilon(p)$  for  $p \in N(\Sigma_1)$ .

**Remark 1.6.** A careful construction of the neighborhood  $\mathcal{N}(\Sigma)$  as well as the normalizations of the contact forms required to perform the convex gluing used to define the Liouville connect sum will be described in Section 3.

**Definition 1.7.** In the above notation, we say that the contact manifold

$$\#_{((\Sigma, \beta), (i_1, i_2))}(M, \xi) := (\#_{(\Sigma, (i_1, i_2))}M, \#_{((\Sigma, \beta), (i_1, i_2))}\xi)$$

is the Liouville connect sum of  $(M, \xi)$  along the Liouville hypersurfaces  $i_1(\Sigma)$  and  $i_2(\Sigma)$ .

When the embeddings  $i_1$  and  $i_2$  of Definition 1.7 are understood, we will use the short-hand notation  $\#_{(\Sigma, \beta)}(M, \xi)$  for  $\#_{((\Sigma, \beta), (i_1, i_2))}(M, \xi)$ . It should be noted that the Liouville connect sum depends on the embeddings  $i_1$  and  $i_2$ , not just the images of  $\Sigma$  under these mappings.

**Example 1.8.** Consider the disjoint union  $(M, \xi) \sqcup (S^{2n+1}, \xi_{std})$  of some arbitrary  $(2n+1)$ -dimensional contact manifold and the standard  $(2n+1)$ -sphere. Let  $L$  be an isotropic  $k$ -sphere in  $(M, \xi)$  with trivial normal bundle. By considering  $S^{2n+1} \subset \mathbb{R}^{2n+2}$  as in Example 1.2, we may define  $L' \subset (S^{2n+1}, \xi_{std}) = \partial(\mathbb{D}^{2n+2}, \lambda_{std})$  to be the isotropic  $k$ -sphere  $S^{2n+1} \cap \text{Span}(x_1, \dots, x_{k+1})$ . Then we can find Liouville hypersurfaces  $(\Sigma_1, \lambda_{std} - \lambda_{can}) \subset (M, \xi)$  and  $(\Sigma_2, \lambda_{std} - \lambda_{can}) \subset (S^{2n+1}, \xi_{std})$  which deformation retract onto  $L$  and  $L'$ , respectively, as described in Example 1.5. Now let  $(\Sigma, \beta)$  be an additional copy of a neighborhood of the zero section of the bundle  $\mathbb{R}^{2m} \oplus T^*S^k \rightarrow S^k$  with  $\beta = \lambda_{std} - \lambda_{can}$  and define Liouville embeddings  $i_1 : (\Sigma, \beta) \rightarrow (M, \xi)$  and  $i_2 : (\Sigma, \beta) \rightarrow (S^{2n+1}, \xi_{std})$  for which  $i_j((\Sigma, \beta)) = (\Sigma_j, \lambda_{std} - \lambda_{can})$ ,  $j = 1, 2$ .

Applying a Liouville connect sum, we have that  $\#_{(\Sigma, \beta)}((M, \xi) \sqcup (S^{2n+1}, \xi_{std}))$  is the same contact manifold as is described by a Weinstein handle attachment along  $L \subset (M, \xi)$  (with respect to some framing of the symplectic normal bundle of  $L$ ). See Section 2.3.

The main result of this paper is the following theorem, whose proof appears in Section 7.

**Theorem 1.9.** *Let  $(M, \xi)$  be a closed, possibly disconnected,  $(2n+1)$ -dimensional contact manifold. Suppose that there are two Liouville embeddings  $i_1, i_2 : (\Sigma, \beta) \rightarrow (M, \xi)$  with disjoint images. Then there is an exact symplectic cobordism  $(W, \lambda)$  whose negative boundary is  $(M, \xi)$  and whose positive boundary is  $\#_{(\Sigma, \beta)}(M, \xi)$ . Moreover, if  $(\Sigma, \beta)$  admits a Stein structure, then so does the cobordism  $(W, \lambda)$ .*

The proof of Theorem 1.9 consists of attaching a symplectic handle  $(H_\Sigma, \omega_\beta)$  to the symplectization of  $(M, \xi)$ . Note that the cobordism  $(W, \lambda)$  described above is always Stein when  $\dim(M) = 3$  as every 2-dimensional Liouville domain admits a Stein structure. The proof of Theorem 1.9 provides an explicit Weinstein handle decomposition of the cobordism  $(W, \lambda)$  in the event that  $(\Sigma, \beta)$  admits the structure of a Stein domain.

When the components of  $(M, \xi)$  appear as convex boundary components of a weak symplectic cobordism  $(W, \omega)$  (see Definition 2.6) and  $i_j : (\Sigma, \beta) \rightarrow (M, \xi)$ ,  $j = 1, 2$ , are Liouville embeddings, then it is possible to attach a slightly modified version of  $(H_\Sigma, \omega_\beta)$  to  $\partial W$  as above, provided the vanishing of a cohomological obstruction. This obstruction always vanishes when  $\dim(M) = 3$ . See Section 7.3.

**Example 1.10.** Consider the Liouville connect sum  $\#_{(\Sigma, \beta)}((M, \xi) \sqcup (S^{2n+1}, \xi_{std}))$  from Example 1.8. In this case, the cobordism  $(W, \lambda)$  from  $((M, \xi) \sqcup (S^{2n+1}, \xi_{std}))$  to  $\#_{(\Sigma, \beta)}((M, \xi) \sqcup (S^{2n+1}, \xi_{std}))$  described in Theorem 1.9 is the same as the usual cobordism (from  $(M, \xi)$  to  $\#_{(\Sigma, \beta)}((M, \xi) \sqcup (S^{2n+1}, \xi_{std}))$ ) provided by a Weinstein handle attachment with a Darboux ball  $(\mathbb{D}^{2n+2}, \lambda_{std})$  removed from its interior. See Sections 2.3 and 7.

**1.5. Applications.** Now we state some consequences of Theorem 1.9 whose proofs will appear later in the text. Again, we will be using the definitions and notation of Section 2.2.

**1.5.1. Open books and fillability monoids.** Our first application of Theorem 1.9 is to the study of contact manifolds determined by open books.

**Definition 1.11.** *Let  $\Sigma$  be an compact, oriented manifold with non-empty boundary. Let  $\text{Diff}^+(\Sigma, \partial\Sigma)$  be the group of orientation preserving diffeomorphisms of  $\Sigma$  which restrict to the identity on some collar neighborhood of  $\partial\Sigma$ . When  $\Sigma$  admits a symplectic form  $\omega$ , the symplectomorphism group of  $(\Sigma, \omega)$  will refer to the subgroup  $\text{Symp}((\Sigma, \omega), \partial\Sigma)$  of  $\text{Diff}^+(\Sigma, \partial\Sigma)$  whose elements preserve  $\omega$ .*

For each pair  $(\Sigma, \Phi)$  with  $\Phi \in \text{Diff}^+(\Sigma, \partial\Sigma)$  we can build a smooth manifold  $M_{(\Sigma, \Phi)}$  defined by

$$M_{(\Sigma, \Phi)} = (\Sigma \times [0, 1]) / \sim \quad \text{where} \\ (\Phi(x), 1) \sim (x, 0) \quad \forall x \in W \quad \text{and} \quad (x, t) \sim (x, t') \quad \forall (x, t), (x, t') \in (\partial W) \times [0, 1].$$

The manifold  $M_{(\Sigma, \Phi)}$  is called the *open book* associated to the pair  $(\Sigma, \Phi)$ . The diffeomorphism class of  $M_{(\Sigma, \Phi)}$  depends only on  $\Phi$  up to conjugation and isotopy in  $\text{Diff}^+(\Sigma, \partial\Sigma)$ . Each  $\Sigma \times \{t\} \subset M_{(\Sigma, \Phi)}$  is called a *page* of the open book. The codimension two submanifold  $\partial\Sigma$  of  $M_{(\Sigma, \Phi)}$  is called the *binding* of the open book and is naturally oriented as the boundary of a page. The diffeomorphism  $\Phi$  is called the *monodromy* of the open book.

**Definition 1.12.** Let  $(M, \xi)$  be a  $(2n + 1)$ -dimensional contact manifold, let  $(\Sigma, \beta)$  be a  $2n$ -dimensional Liouville domain, and let  $\Phi \in \text{Symp}((\Sigma, d\beta), \partial\Sigma)$ . We say  $(M, \xi)$  is supported by the pair  $((\Sigma, \beta), \Phi)$  if  $M = M_{(\Sigma, \Phi)}$  and there is a contact form  $\alpha$  for  $(M, \xi)$  such that

- (1) the Reeb vector field  $R_\alpha$  is positively transverse to the interior of each page of  $M_{(\Sigma, \Phi)}$ ,
- (2) each page of the open book - with a collar neighborhood of its boundary removed - is a Liouville hypersurface in  $(M, \xi)$  which is Liouville homotopic to  $(\Sigma, \beta)$ , and
- (3)  $R_\alpha$  is tangent to the binding.

**Theorem 1.13** ([Gi02, TW75]). Let  $(\Sigma, \beta)$  be a Liouville domain and let  $\Phi \in \text{Symp}((\Sigma, d\beta), \partial\Sigma)$ . Then

- (1)  $M_{(\Sigma, \Phi)}$  naturally carries a contact structure  $\xi_{((\Sigma, \beta), \Phi)}$  supported by the pair  $((\Sigma, \beta), \Phi)$ .
- (2)  $(M_{(\Sigma, \Phi)}, \xi_{((\Sigma, \beta), \Phi)})$  depends only on  $(\Sigma, \beta)$  and  $\Phi$  up to conjugacy and isotopy in  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$ .  
Moreover,
- (3) every 3-dimensional contact manifold is supported by an open book, and
- (4) two contact 3-manifolds  $(M_{(\Sigma, \Phi)}, \xi_{((\Sigma, \beta), \Phi)})$  and  $(M_{(\Sigma', \Psi)}, \xi_{((\Sigma', \beta'), \Psi)})$  are diffeomorphic if and only if the pairs  $(\Sigma, \Phi)$  and  $(\Sigma', \Psi)$  are related by a sequence of positive stabilizations.

For simplicity, we will denote the contact manifold  $(M_{(\Sigma, \Phi)}, \xi_{((\Sigma, \beta), \Phi)})$  described in Theorem 1.13(1) by  $(M, \xi)_{((\Sigma, \beta), \Phi)}$ .

**Remark 1.14.** Giroux [Gi02] has also outlined a program for characterizing high dimensional contact manifolds in terms of open books with Stein pages. When  $\dim(\Sigma) = 2$ , a symplectic form on  $\Sigma$  is simply a volume form and every such  $\Sigma$  admits the structure of a Liouville domain  $(\Sigma, \beta)$ . In this case, after having specified such a 1-form  $\beta$  on  $\Sigma$ , every  $\Phi \in \text{Diff}^+(\Sigma, \partial\Sigma)$  is isotopic to an element of  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$ . Furthermore, any two Liouville 1-forms on a compact oriented surface with boundary are homotopic in the sense of Section 2.2. Therefore the study of monodromies of open books determining contact 3-manifolds reduces to the study of mapping class groups of surfaces.

**Definition 1.15.** Let  $(\Sigma, \beta)$  be a  $2n$ -dimensional Liouville domain. A property  $\mathcal{P}$  of contact  $(2n+1)$ -manifolds is a monoid property for  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$  if the collection of  $\Phi \in \text{Symp}((\Sigma, d\beta), \partial\Sigma)$  for which  $(M, \xi)_{((\Sigma, \beta), \Phi)}$  satisfies  $\mathcal{P}$  is a monoid in  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$ .

**Theorem 1.16.** Let  $(\Sigma, \beta)$  be a Liouville domain.

- (1) “Symplectically fillable” and “exactly symplectically fillable” are monoid properties for  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$ .
- (2) If  $\dim(\Sigma) = 2$ , then “weakly fillable” and “Stein fillable” are monoid properties for  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$ .
- (3) Moreover, if  $(\Sigma, \beta)$  is of any even dimension and admits the structure of a Stein domain, then “Stein fillable” is a monoid property for  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$ .

Theorem 1.16 was motivated by and generalizes results of Baker-Etnyre-van Horn-Morris [BEV10, §1.2] and Baldwin [Ba10, Theorems 1.1-1.3].

The question of whether or not “weakly fillable” is a monoid property for Liouville domains of dimension greater than two appears to be more subtle. Observe that given such a Liouville domain  $(\Sigma, \beta)$  and some  $\Phi \in \text{Symp}((\Sigma, \beta), \partial\Sigma)$ , there is a natural Liouville embedding  $i_\Phi$  of  $(\Sigma, \beta)$  into the contact manifold  $(M, \xi)_{((\Sigma, \beta), \Phi)}$  whose image is the page of the associated open book. The naturality of this embedding follows from the fact that the manifold  $M_{(\Sigma, \Phi)}$  is defined constructively.

**Theorem 1.17.** *Let  $(\Sigma, \beta)$  be a Liouville domain for which  $\dim(\Sigma) > 2$ , and let  $\Phi, \Psi \in \text{Symp}((\Sigma, \beta), \partial\Sigma)$ . Suppose that  $(M, \xi)_{((\Sigma, \beta), \Phi)}$  and  $(M, \xi)_{((\Sigma, \beta), \Psi)}$  admit weak symplectic fillings  $(W_1, \omega_1)$  and  $(W_2, \omega_2)$ , respectively. Then, if  $i_\Phi^* \omega_1 = i_\Psi^* \omega_2$  in  $H^2(\Sigma; \mathbb{R})$ , the contact manifold  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  is weakly symplectically fillable. In particular, if  $H^2(\Sigma; \mathbb{R}) = 0$ , then “weakly symplectically fillable” is a monoid property for  $\text{Symp}((\Sigma, \beta), \partial\Sigma)$ .*

If  $(\Sigma, \beta)$  is a Liouville domain, then  $\text{id}_\Sigma \in \text{Symp}((\Sigma, d\beta), \partial\Sigma)$  is an element of the “exactly symplectically fillable” monoid in  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$ . This is a consequence of the fact that the contact manifold  $(M, \xi)_{((\Sigma, \beta), \text{id}_\Sigma)}$  can be realized as the boundary of the Liouville domain obtained by rounding the corners of  $(\Sigma \times \mathbb{D}^2, \beta + \lambda_{std})$ . Similarly, if  $(\Sigma, \beta)$  admits the structure of a Stein domain, then  $\text{id}_\Sigma$  is an element of the “Stein fillable” monoid in  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$ . This is a consequence of the fact that  $(\Sigma \times \mathbb{D}^2, \beta + \lambda_{std})$  admits the structure of a Stein domain after rounding the corners of the product. For more information on Stein domains of this type, see [C02].

The proof of Theorem 1.16 consists of constructing  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  from  $(M, \xi)_{((\Sigma, \beta), \Phi)} \sqcup (M, \xi)_{((\Sigma, \beta), \Psi)}$  by a Liouville connect sum and then appealing to the existence of the symplectic cobordism  $(W, \lambda)$  described in Theorem 1.9. The fact that this cobordism is exact provides us with the following easy corollary:

**Corollary 1.18.** *Let  $(\Sigma, \beta)$  be a Liouville domain. Then “non-vanishing contact homology with rational coefficients” is a monoid property for  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$ .*

This result is analogous to a theorem first proved in [Ba08, Theorem 1.2] (see also [BEV10, Ba10]) regarding the non-vanishing of contact classes in the Heegaard Floer homologies of contact 3-manifolds.

**1.5.2. Fillability of fibered contact manifolds.** In Section 8.3 we define a family  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$  of  $(2n + 1)$ -dimensional contact manifolds which fiber over the circle  $S^1$ , each determined by a  $(2n)$ -dimensional Liouville domain  $(\Sigma, \beta)$  and a pair of symplectomorphisms  $\Phi, \Psi \in \text{Symp}((\Sigma, d\beta), \partial\Sigma)$ .

More specifically, consider a tubular neighborhood  $[-1, 1] \times \partial\mathcal{N}(\Sigma)$  where  $\mathcal{N}(\Sigma)$  is the model neighborhood described in Section 1.2 and  $\theta$  is a coordinate on  $[-1, 1]$ . The manifold  $[-1, 1] \times \mathcal{N}(\Sigma)$  inherits a  $\theta$ -invariant contact structure from the contact form  $dz + \beta$  on  $N(\Sigma)$ . By gluing

- (1) the positive region of  $\{1\} \times \partial N(\Sigma)$  to the negative region of  $\{-1\} \times \partial\mathcal{N}(\Sigma)$  using the map  $\Phi$  and
- (2) the positive region of  $\{-1\} \times \partial N(\Sigma)$  to the negative region of  $\{1\} \times \partial\mathcal{N}(\Sigma)$  using the map  $\Psi$

we obtain  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$ . See Figure 2. In the simplest case, with  $\Phi = \Psi = \text{id}_\Sigma$ ,  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$  is the boundary of the Liouville domain obtained by rounding the corners of  $(\Sigma \times \mathbb{D}^* S^1, \beta - \lambda_{can})$ .

When  $\dim(\Sigma) = 2$ , this family of contact manifold forms a subset of the collection of universally tight surface bundles over  $S^1$ . The tightness and fillability of contact structures on surface bundles over the circle have been studied extensively. See, for example, [DG01, EI96, Gi94, Hon00b, HKM03, VHM07, Wen10].

A slight variation of the proof of Theorem 1.16 gives the following:

**Theorem 1.19.** *Let  $(\Sigma, \beta)$  be a  $2n$ -dimensional Liouville domain.*

- (1) *If  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  is symplectically (exactly) fillable, then  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$  is also symplectically (exactly) fillable.*
- (2) *If  $(\Sigma, \beta)$  admits a Stein structure, and  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  is Stein fillable then so is  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$ .*
- (3) *If  $\dim(M) = 3$  and  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  is weakly fillable, then so is  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$ .*

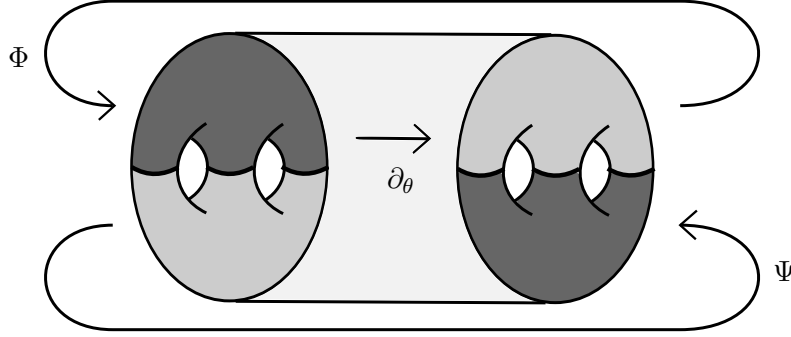


FIGURE 2. The contact manifold  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$ . It is determined by the convex gluing instructions shown on the boundary of the contact manifold  $([-1, 1] \times \partial\mathcal{N}(\Sigma), \xi_{(\Sigma, \beta)})$ . See Section 8.3.1 for further explanation.

Furthermore, if  $\dim(M) > 3$ ,  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  admits a weak filling  $(W, \omega)$ , and  $i_{\Phi \circ \Psi}^* \circ (id_{\Sigma} - \Phi^*)\omega \in \Omega^2(\Sigma)$  is exact, then  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$  is also weakly fillable.

The Liouville embedding  $i_{\Phi \circ \Psi} : (\Sigma, \beta) \rightarrow (M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  in the above theorem is as described in the discussion preceding the statement of Theorem 1.17.

**1.5.3. Fillability of branched covers.** Our next application of Theorem 1.9 concerns branched covers of contact manifolds. Let  $(C, \zeta) \subset (M, \xi)$  be a null-homologous, codimension two contact submanifold of the contact manifold  $(M, \xi)$  and denote by  $(M, \xi)_{C, q}$  the associated  $q$ -fold cyclic branched cover of  $(M, \xi)$ , branched over  $(C, \zeta)$  for a positive integer  $q$ . The contact manifold  $(M, \xi)_{C, q}$  is described in Proposition 8.18.

**Theorem 1.20.** *Let  $(M, \xi)$  be a  $(2n + 1)$ -dimensional contact manifold. Let  $(C, \zeta) \subset (M, \xi)$  be a codimension two contact submanifold bounding a Liouville hypersurface  $(\Sigma, \beta) \subset (M, \xi)$ . Then for  $q \geq 2$  there is an exact symplectic cobordism  $(W, \lambda)$  whose negative boundary is  $\sqcup^q(M, \xi)$  and whose positive boundary is  $(M, \xi)_{C, q}$ . If  $(\Sigma, \beta)$  admits a Stein structure, then so does the cobordism  $(W, \lambda)$ . Moreover, if  $(M, \xi)$  is weakly fillable, then so is  $(M, \xi)_{C, q}$ .*

Theorem 1.20 is similar in flavor to results of Baldwin [Ba10] and Harvey-Kawamuro-Plamenevskaya [HPK09] regarding branched coverings of contact 3-manifolds which we will summarize in Theorem 8.20.

**1.5.4. Liouville domains without Stein structures.** In Section 8.6 we discuss how Theorem 1.9 can be used to construct Liouville domains and exact symplectic cobordisms which do not admit Stein structures. The examples we provide have connected boundary, although their construction relies on the existence of Liouville domains with disconnected boundary – c.f. [Ge94, MNW11, Mc91]. The examples appearing in Section 8.6 show that the cobordisms described in Theorem 1.9 are not always Stein.

**1.5.5. Contact  $(1/k)$ -surgery and squares of generalized Dehn twists.** In [DG01], Ding-Geiges define contact  $(1/k)$ -surgery along Legendrian knots in contact 3-manifolds, generalizing Weinstein’s Legendrian surgery [Wei91] in the 3-dimensional case. Using the Liouville connect sum and generalized Dehn twists [Ar95, S97], we provide a definition of contact  $(1/k)$ -surgery along Legendrian  $n$ -spheres in contact  $(2n + 1)$ -manifolds for arbitrary  $n \geq 1$ . For  $k = -1$ , our definition coincides with that Legendrian surgery. For  $n = 1$ , our definition coincides with the usual notion of contact  $(1/k)$ -surgery. This construction is described in Section 9. There we observe that many properties of contact  $(1/k)$ -surgery known to hold in the 3-dimensional case easily carry over to contact manifolds of arbitrary dimension.

A theorem of Seidel asserts that the square of a generalized Dehn twist along  $S^2$  is smoothly isotopic to the identity mapping. In Section 9.2 we make use of his proof [S99, Lemma 6.3] as well as some homotopy theoretic results of Adams [Ad58] and James-Whitehead [JW54] to enhance this theorem.

**Theorem 1.21.** *Denote by  $\tau_n \in \text{Symp}((\mathbb{D}^*S^n, -d\lambda_{\text{can}}), \partial\mathbb{D}^*S^n)$  a generalized Dehn twist along the  $n$ -sphere for  $n > 0$ . Then  $\tau_n^2$ , considered as an element of  $\text{Diff}^+(\mathbb{D}^*S^n, \partial\mathbb{D}^*S^n)$ , is isotopic to the identity mapping if and only if  $n$  is either 2 or 6.*

**Remark 1.22.** *A considerably stronger version of this result in the case  $n = 2$  can be found in [S97, Proposition 2.6]. As remarked there, this stronger statement is specific to the case  $n = 2$ .*

With the help of Theorem 1.21, we construct exotic contact structures on  $S^5$  and  $S^{13}$  by performing contact  $(1/2k)$ -surgeries along Legendrian spheres in  $(S^5, \xi_{\text{std}})$  and  $(S^{13}, \xi_{\text{std}})$ . The following theorem then immediately follows from this construction.

**Theorem 1.23.** *When considered as elements of  $\text{Symp}((\mathbb{D}^*S^n, -\lambda_{\text{std}}), \partial\mathbb{D}^*S^n)$  ( $n = 2, 6$ ),  $\tau_2^2$  and  $\tau_6^2$  are not isotopic to the identity.*

The case  $n = 2$  of Theorem 1.23 was originally proved in [S97] using Floer homology. Our proof is a consequence of the exoticity of the contact spheres described above, which we establish using the Eliashberg-Floer-Gromov-McDuff theorem [El91, Mc91] asserting that an exact symplectic filling  $(W, \lambda)$  of  $(S^{2n+1}, \xi_{\text{std}})$  must be such that  $W$  is diffeomorphic to  $\mathbb{D}^{2n+2}$ .

We note that the same line of reasoning we use to prove Theorem 1.23 can be applied to ordinary Legendrian surgeries or the “overtwisted” spheres appearing in Bourgeois-van Koert [BK10] to obtain Theorem 1.23. This will be further discussed in Section 9. More examples of exotic contact structures on spheres of dimension greater than three can be found in [DG04], [El91], and [U99]. Non-standard contact structures on  $S^3$  are completely understood by [Be82], [El89], and [El92]. See also [Hu11]. Another contact-geometric proof of the symplectic non-triviality of squares of Dehn twists along  $S^2$  – obtained by analyzing the contact manifolds described in [U99] – can be found in [KN05].

**1.6. Outline.** The remainder of this paper is organized as follows:

**Section 2.** This section consists mostly of the establishment of notation and includes a brief overview of Weinstein handle attachment.

**Section 3.** In this section we carry out the technical details concerning neighborhood theorems for Liouville hypersurfaces required to rigorously define the Liouville connect sum. In Section 3.6 we discuss how the gluing map defining the Liouville connect sum can be modified by symplectomorphisms.

**Section 4.** Here we analyze neighborhoods of Liouville submanifolds of codimension greater than one. The main result is Theorem 4.5, which essentially states that there is no loss of generality in only considering Liouville submanifolds of codimension one. This section also contains results concerning the existence of contact and Liouville 1-forms on the total spaces of symplectic disk bundles which may be of independent interest.

**Section 5.** In this section we give various examples of Liouville hypersurfaces in contact manifolds. We also describe a general method of constructing embeddings of contact 3-manifolds into contact 5-manifolds which cannot bound Liouville hypersurfaces.

**Section 6.** This section describes some more basic consequences of Definition 1.3, such as some results which equate overtwistedness of contact 3-manifolds with the existence of Liouville surfaces satisfying certain properties.

**Section 7.** This section contains the proof of Theorem 1.9. We also discuss when it is possible to attach a modified version of the cobordism  $(W, \lambda)$  to the positive boundary of a weak symplectic cobordism.



**Section 8.** Here we prove some of the corollaries of Theorem 1.9 stated in Section 1.5. We also provide an example which shows how the proof of Theorem 1.9 can be used to draw a Kirby diagrams for the cobordism described in Theorem 1.9 in the event that the contact manifold  $(M, \xi)$  is 3-dimensional.

**Section 9.** This section defines and outlines some of the basic properties of contact  $(1/k)$ -surgery. There we also briefly review known facts about contact  $(1/k)$ -surgery on contact 3-manifolds and generalized Dehn twists.

**Section 10.** In this section we use open books and contact  $(1/k)$ -surgery to study the symplectic topology of generalized Dehn twists, proving Theorems 1.21 and 1.23.

## 2. NOTATION AND DEFINITIONS

Here we outline some prerequisite material. This section begins by establishing some notation which will be used throughout the paper. Then in Section 2.2 we recall some definitions and basic examples from symplectic geometry. In Section 2.3 we provide a brief overview handlebody constructions and decompositions of Stein domains and cobordisms.

**2.1. Notation.** Suppose that  $X$  is a smooth  $n$ -dimensional manifold.

A vector bundle  $E \rightarrow X$  over  $X$  will always assumed to be smooth and have finite rank. Such a vector bundle will always be considered as a real vector bundle, even if it is equipped with a complex structure.

The cotangent bundle will be written  $T^*X$ . After equipping  $X$  with a Riemannian metric  $\langle \cdot, \cdot \rangle$  we may consider the unit disk and sphere bundles in  $T^*X$ . These will be denoted by  $\mathbb{D}^*X$  and  $S^*X$ , respectively. In this paper, we will not be interested in the geometry of any particular Riemannian metrics, and so – with the exception of Section 9.2 – we will refer to  $\mathbb{D}^*X$  and  $S^*X$  without explicitly specifying a metric.

For a vector field  $V$  on  $X$ , the diffeomorphism of  $X$  determined by the time- $t$  flow of  $V$  will be written  $\exp(t \cdot V)$ . Lie derivatives with respect to  $V$  will be written  $\mathcal{L}_V$ .

If  $X$  is closed and oriented, then the fundamental class of  $X$  in  $H_n(X; \mathbb{Z})$  will be written  $[X]$ .

For a contact manifold  $(M, \xi)$ , with contact form  $\alpha$ , the associated *Reeb vector field* will be denoted by  $R_\alpha$ . Recall that  $R_\alpha$  is uniquely determined by the equations

$$\alpha(R_\alpha) = 1 \quad \text{and} \quad d\alpha(R_\alpha, \cdot) = 0.$$

For a smooth function  $f \in C^\infty(W)$  on a symplectic manifold  $(W, \omega)$ , the *Hamiltonian vector field*  $X_f$  is defined by the convention that  $df(\cdot) = \omega(X_f, \cdot)$ .

**2.2. Some definitions from symplectic geometry.** We continue with the discussion started in Section 1.1.

**Example 2.1.** Let  $X$  be a closed, smooth  $n$ -manifold. The cotangent bundle  $T^*X$  of  $X$  admits a Liouville 1-form  $-\lambda_{can}$ . If  $(q_1, \dots, q_n)$  is a coordinate chart on  $X$  then, in the associated coordinate  $(q_1, \dots, q_n, p_1, \dots, p_n)$  on  $T^*X$  – the  $p_j$  being the coefficients of the  $dq_j$  – we have  $\lambda_{can} = \sum p_j dq_j$ . The associated Liouville vector field is given by the radial vector field, written  $X_{can} = \sum p_j \partial_{p_j}$  in local coordinates. Then  $(\mathbb{D}^*X, -\lambda_{can})$  is a Liouville domain. The induced contact structure  $\xi_{can} = \text{Ker}(-\lambda_{can})$  on  $S^*X$  is called the *canonical contact structure*. Similarly,  $\lambda_{can}$  is called the *canonical 1-form* and  $-d\lambda_{can}$  is called the *canonical symplectic form* on  $T^*X$ . Note that  $\xi_{can}$  is independent of the metric used to define  $S^*X$  by Gray's stability theorem.

**Example 2.2.** Let  $(\Sigma, \beta)$  and  $(\Sigma', \beta')$  be two Liouville domains. Then  $(\Sigma \times \Sigma', \beta + \beta')$  admits the structure of a Liouville domain after rounding the corners  $(\partial\Sigma) \times (\partial\Sigma')$  of the product.

**Definition 2.3.** A Stein domain  $(\Sigma, \beta, J)$  is a Liouville domain  $(\Sigma, \beta)$  equipped with an integrable complex structure  $J$  such that:

- (1)  $\beta = -df \circ J$  for some  $f \in C^\infty(W, \mathbb{R})$ , and
- (2)  $\partial\Sigma = f^{-1}(c)$  for some regular value  $c \in \mathbb{R}$  of the function  $f$ .

Note that  $(\mathbb{D}^{2n+2}, \lambda_{std}, J)$ , as described in Example 1.2 is Stein with  $J$  being the standard complex structure and  $f(z) = \|z\|^2$ .

**Definition 2.4.** Let  $(M, \xi)$  be a contact manifold and suppose that  $(M', \xi')$  is another contact manifold with  $\dim(M') = \dim(M)$ . A symplectic manifold  $(W, \omega)$  with  $\partial W = M \sqcup (-M')$  is

- (1) a symplectic cobordism from  $(M, \xi)$  to  $(M', \xi')$  if both  $M$  and  $M'$  are contact-type hypersurfaces in  $(W, \omega)$  and the induced contact structures on  $M$  and  $M'$  are  $\xi$  and  $\xi'$  respectively.
- (2) an exact symplectic cobordism from  $(M, \xi)$  to  $(M', \xi')$  if it is a symplectic cobordism, and the 1-form  $\lambda$  used to identify  $M$  and  $M'$  as contact-type hypersurface (as described in Section 1.1) is defined on all of  $W$ .
- (3) a Stein cobordism from  $(M, \xi)$  to  $(M', \xi')$  if it is an exact symplectic cobordism and there is an integrable complex structure  $J$  on  $W$  for which  $\lambda = -df \circ J$  for a function  $f \in C^\infty(W)$  such that  $M$  and  $M'$  are inverse images of regular values of  $f$ .

We call  $(M, \xi)$  the convex boundary of  $W$  and  $(M', \xi')$  the concave boundary of cobordism  $(W, \omega)$ . Similarly we say that  $(M, \xi)$  is symplectically (resp. exactly, Stein) fillable if there is a symplectic (resp. exact, Stein) cobordism from  $(M, \xi)$  to the empty manifold.

When a symplectic cobordism  $(W, \omega)$  is exact (or Stein) with  $\lambda \in \Omega^1(W)$  satisfying  $d\lambda = \omega$  as in item (2) of the above definition, it shall be specified by the pair  $(W, \lambda)$  to emphasize exactness.

Two exact symplectic cobordisms  $(W, \lambda)$  and  $(W, \lambda')$  will be called *homotopic* if there is a smooth  $[0, 1]$ -family  $\lambda_t$  of 1-forms on  $W$  such that  $\lambda_0 = \lambda$ ,  $\lambda_1 = \lambda'$ , and  $(W, \lambda_t)$  is an exact symplectic cobordism for all  $t \in [0, 1]$ . Note that if  $(W, \lambda)$  and  $(W, \lambda')$  are homotopic, then the concave and convex boundaries of  $(W, \lambda)$  and  $(W, \lambda')$  are pairwise contact-diffeomorphic.

**Example 2.5.** Let  $(M, \xi)$  be a closed contact manifold with contact 1-form  $\alpha$ . Then the *compact symplectization*  $([1/2, 1] \times M, t \cdot \alpha)$  is an exact symplectic cobordism from  $(M, \xi)$  to itself. Writing  $t$  for a coordinate on  $[1/2, 1]$ , we say that an almost complex structure  $J$  on  $[1/2, 1] \times M$  is *adapted to the symplectization* if it satisfies

- (1)  $J(\xi) = \xi$ ,
- (2)  $J(t\partial_t) = R_\alpha$ , and
- (3)  $J$  is  $t$ -invariant.

Provided such a  $J$ , it can be seen that  $([1/2, 1] \times M, t \cdot \alpha)$  admits the structure of a Stein cobordism with the function described in Definition 2.3 being  $f = \frac{1}{2}t^2$ .

In this paper, we will not be concerned with complex structures associated to Stein domains and cobordisms. Our primary reason for studying Stein domains and cobordisms is that they admit special handle decompositions which will be described in the next section. Accordingly, we shall say that a Liouville domain  $(\Sigma, \beta)$  (exact cobordism  $(W, \lambda)$ ) is *Stein* if it is homotopic to a Liouville domain  $(\Sigma, \beta')$  (resp. exact cobordism  $(W, \lambda')$ ) which admits a complex structure  $J$  such that  $(\Sigma, \beta', J)$  (or  $(W, \lambda', J)$ ) is a Stein domain (resp. cobordism). By the same reasoning, whenever we speak of the dimension  $\dim(W)$  of a Stein manifold  $(W, \lambda, J)$ , we will always be referring to its dimension  $\dim_{\mathbb{R}}(W)$  as a real, smooth manifold.

There is one last type of cobordism we will consider in this paper:

**Definition 2.6.** Let  $(M, \xi)$  and  $(M', \xi')$  be  $(2n + 1)$ -dimensional contact manifolds. A compact symplectic manifold  $(W, \omega)$  is a weak symplectic cobordism from  $(M, \xi)$  to  $(M', \xi')$  if

- (1)  $\partial W = M \sqcup (-M')$ ,
- (2) both  $\alpha \wedge (d\alpha + \omega|_\xi)^n$  and  $\alpha \wedge \omega|_{\xi'}^n$  define positive volume forms on  $M$  for every choice of contact 1-form  $\alpha$  for  $(M, \xi)$ , and

- (3) both  $\alpha' \wedge (d\alpha' + \omega|_{\xi'})^n$  and  $\alpha' \wedge \omega|_{\xi'}^n$  define positive volume forms on  $M'$  for every choice of contact 1-form  $\alpha'$  for  $(M', \xi')$ .

In the event that  $M' = \emptyset$ , we say that  $(W, \omega)$  is a weak symplectic filling of  $(M, \xi)$ .

The above definition – first stated in [MNW11] – is related to the notion of an  $\omega$ -dominating cobordism – first defined in [EG89, §3]. We say that a symplectic manifold  $(W, \omega)$  with non-empty boundary *dominates* a contact structure  $\xi$  on its boundary if the conformal class of  $\omega|_{\xi}$  coincides with the conformal class of symplectic structure on  $\xi$  determined by a contact form for  $(\partial W, \xi)$ . This definition is the same as Definition 2.6 for 3-dimensional contact manifolds. See [MNW11] for further discussion.

As pointed out by McDuff in [Mc91, Lemma 2.1],  $\omega$ -dominating cobordisms between (or fillings of) contact manifolds of dimension greater than or equal to 5 are symplectic cobordisms (fillings) in the sense of Definition 2.4(1). However, there are 4-dimensional weak symplectic – but not symplectic – cobordisms and fillings. See, for example, [DG01, Theorem 1], [EI96, §3], [Gi94, §2.D], and [Wen10, §1.2]. In [MNW11] a strategy is described for constructing weakly fillable – but not symplectically fillable – contact manifolds of all dimensions greater than three, with examples provided in dimension five [MNW11, Theorem E].

**2.3. Weinstein handles.** Now we define Weinstein handle attachments and outline their role in the construction of Stein domains. We have included this material as the proof of the second statement of Theorem 1.9, which is contained in Section 7.2, will require an explicit description of the differential forms involved. We also hope that the reader will see a connection between the construction of a Weinstein handle attachment and the construction of the cobordism  $(W, \lambda)$  described in Theorem 1.9. They are essentially the same.

It should be noted that we could alternatively have chosen to state many of the results of this paper in the language of *Weinstein manifolds*. See [EG89] and [EI90].

**2.3.1. Definition and construction of the handle.** Consider  $\mathbb{R}^{2n}$  with its standard Liouville form  $\lambda_{std}$  and Liouville vector field  $X_{\lambda_{std}} = \frac{1}{2} \sum_1^n (x_j \partial_{x_j} + y_j \partial_{y_j})$  as described in Example 1.2. Let  $\mathbb{D}^k \subset \mathbb{R}^{2n}$  be the unit disk in the plane  $\text{Span}(x_1, \dots, x_k)$ . Then  $\mathbb{D}^k$  is an isotropic submanifold of  $(\mathbb{R}^{2n}, d\lambda_{std})$ . Consider a tubular neighborhood  $H_{n,k} := \mathbb{D}^k \times \mathbb{D}^{2n-k}$  of  $\mathbb{D}^k$ . Then

$$\partial H_{n,k} = ((\partial \mathbb{D}^k) \times \mathbb{D}^{2n-k}) \cup (\mathbb{D}^k \times (\partial \mathbb{D}^{2n-k})) = (S^{k-1} \times \mathbb{D}^{2n-k}) \cup (\mathbb{D}^k \times S^{2n-k-1}).$$

Now consider the function  $f_k(x, y) = \sum_1^k x_j y_j$ . The Hamiltonian vector field of  $f_k$  with respect to  $d\lambda_{std}$  is  $X_{f_k} = \sum_1^k (-x_j \partial_{x_j} + y_j \partial_{y_j})$ . Then  $X_{\lambda_{std}} + X_{f_k}$  is a symplectic dilation of  $(\mathbb{R}^{2n}, d\lambda_{std})$  which points into  $H_{n,k}$  along  $S^{k-1} \times \mathbb{D}^{2n-k}$  and out of  $H_{n,k}$  along  $\mathbb{D}^k \times S^{2n-k-1}$ . In other words, the 1-form

$$(2.3.1) \quad \lambda_{n,k}(\ast) := d\lambda_{std}(X_{\lambda} + X_{f_k}, \ast) = \sum_1^k \left( \frac{3}{2} x_j dy_j + \frac{1}{2} y_j dx_j \right) + \frac{1}{2} \sum_{k+1}^n (x_j dy_j - y_j dx_j)$$

determines a contact structure on each of the smooth pieces of  $\partial H_{n,k}$ , such that  $S^{k-1} \times \mathbb{D}^{2n-k}$  is concave and  $\mathbb{D}^k \times S^{2n-k-1}$  is convex.

**Definition 2.7.**  $(H_{n,k}, \lambda_{n,k})$  is called the  $2n$ -dimensional Weinstein  $k$ -handle.

Now suppose that  $(W, \omega)$  is a  $(2n+2)$ -dimensional symplectic cobordism from  $(M', \xi')$  to  $(M, \xi)$  and that there is a contact embedding of  $(S^{k-1} \times \mathbb{D}^{2n+2-k}, \text{Ker}(\lambda_{n+1,k}))$  into  $(M, \xi)$ . By Equation 2.3.1  $L := S^{k-1} \times \{0\}$  is an isotropic submanifold of  $(M, \xi)$ . On a collar neighborhood  $(\frac{1}{2}, 1] \times M$  of  $M$  in  $W$ , we can write  $\omega = d(t \cdot \alpha)$  where  $\alpha$  is a contact form for  $(M, \xi)$ . As  $(M, \xi)$  is a convex component of  $(W, \omega)$  and  $S^{k-1} \times \mathbb{D}^{2n+2-k}$  is a concave component of  $(H_{n+1,k}, \lambda_{n+1,k})$  then we can patch together the Liouville forms  $t \cdot \alpha$  and  $\lambda_k$  on the manifold

$$W \cup_{S^{k-1} \times \mathbb{D}^{2n+2-k}} H_{n+1,k}$$

to get a new symplectic cobordism whose concave boundary is  $(M', \xi')$  and whose convex boundary is a contact manifold  $(M'', \xi'')$  obtained from  $(M, \xi)$  via surgery. This procedure is called *Weinstein handle attachment* along  $L$  [Wei91]. When  $k = n + 1$ , the submanifold  $L$  is Legendrian and the handle attachment is often called *Legendrian surgery*.

**Remark 2.8.** *A more careful description of the gluing map for Weinstein handle attachment can be found in [Wei91] or by following Steps 4 and 5 of the proof of Theorem 1.9 appearing in Section 7.*

**Remark 2.9.** *The fact that Weinstein handle attachment is an example of a Liouville connect sum – as described in Example 1.10 – follows from the fact that after rounding the corners of the handle  $H_{n,k}$ , its boundary is  $(S^{2n-1}, \xi_{std})$ . The attaching locus  $S^{k-1} \times \mathbb{D}^{2n-k}$  then corresponds to the boundary  $\partial\mathcal{N}(\Sigma)$  of the tubular neighborhood  $\mathcal{N}(\Sigma)$  described in Example 1.8.*

**Example 2.10.** *Again consider the Lagrangian disk  $L = \text{Span}(x_1, \dots, x_n) \cap \mathbb{D}^{2n}$  in  $(\mathbb{R}^{2n}, d\lambda_{std})$ . Then  $\partial L$  is a Legendrian sphere in  $(S^{2n-1}, \xi_{std}) = \partial(\mathbb{D}^{2n}, \lambda_{std})$ . Suppose that we attach a Weinstein handle  $H_{n,n}$  to  $\partial\mathbb{D}^{2n}$  along  $\partial L$  producing a new Liouville domain  $(W, \lambda)$ . If we write  $L'$  for the core disk  $\mathbb{D}^n \times \{0\} \subset H_{n,n}$  of the Weinstein handle, we see that  $L \cup L'$  is a closed Lagrangian submanifold of  $(W, d\lambda)$  which is homeomorphic to the sphere  $S^n$ . By applying the time  $t$  flow of the vector field  $-X_\lambda$  on  $W$  (for  $t \in (0, \infty)$  arbitrarily large) and appealing to the Weinstein neighborhood theorem for Lagrangian submanifolds, we see that  $(W, \lambda)$  is Liouville homotopic to the cotangent disk bundle  $(\mathbb{D}^*(L \cup L'), -\lambda_{can})$ .*

**2.3.2. Handle decompositions of Stein domains.** The following theorem, due to Eliashberg [El90], asserts the usefulness of Weinstein handle attachment in the construction of Stein domains and cobordisms.

**Theorem 2.11.** *Let  $(M, \xi)$  and  $(M', \xi')$  be  $(2n + 1)$ -dimensional contact manifolds.*

- (1) *Let  $(W, \lambda)$  be a Stein cobordism with convex end  $(M, \xi)$ . If we attach a Weinstein handle to  $W$  along an isotropic sphere  $L \subset (M, \xi)$ , then the resulting symplectic cobordism is also Stein.*
- (2) *A  $(2n)$ -dimensional Liouville domain  $(W, \lambda)$  is Stein if and only if it admits a filtration*

$$\sqcup(\mathbb{D}^{2n}, \lambda_{std}) = (W_0, \lambda_0) \subset \dots \subset (W_n, \lambda_n) = (W, \lambda)$$

*where each  $(W_k, \lambda_k)$  is a Stein domain obtained from  $(W_{k-1}, \lambda_{k-1})$  by the attachment of a finite number of  $(2n)$ -dimensional Weinstein  $k$ -handles.*

- (3) *Similarly, a  $(2n + 2)$ -dimensional symplectic cobordism  $(W, \omega)$  from  $(M', \xi')$  to  $(M, \xi)$  is Stein if and only if it can be obtained from the compact symplectization of  $(M', \xi')$  by a finite sequence of Weinstein handle attachments.*

The above theorem implies that a concatenation of Stein cobordisms is Stein. This observation will be particularly useful in Section 8. Statements analogous to Theorem 2.11(2) also exist for non-compact Stein manifolds. For further discussion of the topology of Stein manifolds with an emphasis on the 4-dimensional case, see [Gom98], [Gom09], and [OS04].

### 3. NEIGHBORHOOD CONSTRUCTIONS AND CONVEX GLUING

In this section we provide a rigorous account of the construction of neighborhoods of Liouville hypersurfaces and the modifications of contact forms necessary to define the Liouville connect sum. We also briefly discuss general convex hypersurfaces and convex gluing. This will later be useful for the constructions of contact manifolds in Section 8. We begin by stating some results regarding contact forms on tubular neighborhoods of Liouville hypersurfaces.

### 3.1. Neighborhood theorems.

**Lemma 3.1.** *Suppose that  $(\Sigma, \beta)$  is a Liouville hypersurface contained in the interior of a contact manifold  $(M, \xi)$  and that  $\alpha$  is a contact form for  $(M, \xi)$  for which  $\alpha|_{T\Sigma} = \beta$ . Then for a sufficiently small positive constant  $\epsilon$ , there is a neighborhood of  $\Sigma$  of the form*

$$N(\Sigma) = [-\epsilon, \epsilon] \times \Sigma \quad \text{satisfying} \quad \alpha|_{N(\Sigma)} = dz + \beta.$$

Here  $z$  is a coordinate on  $[-\epsilon, \epsilon]$ , and  $\Sigma = \{0\} \times \Sigma$ .

*Proof.* The requirement that  $d\beta$  is symplectic on  $\Sigma$  is equivalent to the condition that the Reeb vector field  $R_\alpha$  for  $\alpha$  is everywhere transverse to  $\Sigma$ . Define a map  $[-\epsilon, \epsilon] \times \Sigma \rightarrow M$  by

$$(z, x) \mapsto \exp(z \cdot R_\alpha)(x).$$

For  $\epsilon > 0$  sufficiently small, this will be an embedding. As  $\alpha$  is  $R_\alpha$ -invariant, it pulls back to  $[-\epsilon, \epsilon] \times \Sigma$  as desired.  $\square$

The remainder of Section 3.1 describes how contact forms can be modified on neighborhoods of Liouville hypersurfaces as described in the previous lemma. Our exposition follows that of [CGH10, §2.2] in which contact forms on contact manifolds supported by open books are analyzed.

**Lemma 3.2.** *Suppose that  $\beta$  and  $\beta'$  are two Liouville forms on a compact manifold  $\Sigma$  which agree on a collar neighborhood of  $\partial\Sigma$  and satisfy  $d\beta = d\beta'$ . Then there is an isotopy  $\phi_t$ ,  $t \in [0, 1]$ , of  $\Sigma$  such that*

- (1)  $\phi_0 = \text{id}_\Sigma$  and  $\phi_1^* \beta - \beta' = df$  for some smooth function  $f$  on  $\Sigma$  which vanishes on a collar neighborhood of  $\partial\Sigma$ ,
- (2) there is a collar neighborhood of  $\partial\Sigma$  on which  $\phi_t$  is the identity mapping for all  $t \in [0, 1]$ , and
- (3)  $\phi_t^* d\beta = d\beta$  for all  $t \in [0, 1]$ .

*Proof.* Define a vector field  $V$  on  $\Sigma$  as the unique solution to the equation  $d\beta(V, *) = \beta - \beta'$ . Then  $\mathcal{L}_V(d\beta) = 0$  and  $\mathcal{L}_V(\beta - \beta') = 0$  so that  $\exp(tV)$  preserves  $d\beta$  and  $\beta - \beta'$ . Moreover,  $\exp(t \cdot V)$  is equal to the identity on a collar neighborhood of  $\partial\Sigma$ . Now we calculate

$$\begin{aligned} \frac{\partial}{\partial t} (\exp(t \cdot V)(\beta)) &= \exp(t \cdot V)^*(\mathcal{L}_V \beta) = \exp(t \cdot V)^*(d\beta(V, *) + d(\beta(V))) \\ &= \exp(t \cdot V)^*(\beta - \beta' + d(\beta(V))) = \beta - \beta' + dg_t, \\ \text{where } g_t &= \exp(t \cdot V)^*(\beta(V)). \end{aligned}$$

It follows that  $\exp(1 \cdot V)^* \beta = \beta' + df$  where  $f = \int_0^1 g_t dt$ . Defining  $\phi_t = \exp(t \cdot V)$ , the proof is complete.  $\square$

**Lemma 3.3.** *Let  $(\Sigma, \beta) \subset (M, \xi)$  be a Liouville hypersurface with neighborhood  $N(\Sigma) = [-\epsilon, \epsilon] \times \Sigma \subset M$  satisfying  $\alpha|_{N(\Sigma)} = dz + \beta$  where  $\alpha$  is a contact form for  $(M, \xi)$ . Suppose that  $\beta'$  is another Liouville 1-form for  $\Sigma$  such that  $\beta - \beta' = df$  for a smooth function  $f$  on  $\Sigma$  which vanishes on a collar neighborhood of  $\partial\Sigma$ . Then there is a family  $\alpha_t$ ,  $t \in [0, 1]$ , of contact forms on  $M$  such that:*

- (1)  $\alpha_0|_{T\Sigma} = \beta$ ,  $\alpha_1|_{T\Sigma} = \beta'$ ,
- (2)  $\alpha_t - \alpha = 0$  for all  $t \in [0, 1]$  on  $M \setminus [-\epsilon, \epsilon] \times \Sigma$ , and
- (3)  $\alpha_t|_{T\Sigma}$  is a Liouville 1-form on  $\Sigma$  for all  $t \in [0, 1]$ .

*Proof.* Let  $b : [-\epsilon, \epsilon] \rightarrow [0, 1]$  be a function satisfying  $b(0) = 1$  and such that  $b$  and all of its derivatives are zero at the points  $\pm\epsilon$ . Define  $f_{z,t} = 1 + t(c + b \cdot f)$  to be a  $[0, 1]$ -family of functions in  $C^\infty([-\epsilon, \epsilon] \times \Sigma)$

where  $c$  is a constant chosen so that  $f_{z,t} > 0$  for all  $z \in [-\epsilon, \epsilon]$  and  $t \in [0, 1]$ . Now define a  $[0, 1]$ -family of 1-forms on  $N(\Sigma)$  by

$$\alpha_t = \frac{f_{z,t}dz + t \cdot b(z)df}{1 + tc} + \beta.$$

It follows that

- (1)  $\alpha_0 = \alpha$ ,
- (2)  $\alpha_t$  and all of its derivatives agree with  $\alpha$  on  $\partial([-\epsilon, \epsilon] \times \Sigma)$  and so can be extended to  $M \setminus ([-\epsilon, \epsilon] \times \Sigma)$  by  $\alpha$ ,
- (3)  $d\alpha_t = d\beta$  for all  $t$ ,
- (4)  $\alpha_t$  is a contact form for all  $t$ , and
- (5)  $\alpha_t|_{T\Sigma}$  is a Liouville 1-form on  $\Sigma$  for all  $t$  with  $\alpha_1|_{T\Sigma} = \beta' + df$ .

This completes the proof.  $\square$

**3.2. Construction of  $\mathcal{N}(\Sigma)$ .** Now we give a rigorous description of the edge-rounding on  $N(\Sigma) = [-\epsilon, \epsilon] \times \Sigma$ , producing a model neighborhood  $\mathcal{N}(\Sigma)$  of  $\Sigma$  with smooth, convex boundary.

Due to the Weinstein neighborhood theorem for contact type hypersurfaces in symplectic manifolds, we can decompose  $\Sigma$  into two parts  $\Sigma = \widehat{\Sigma} \cup C$ . Here  $\widehat{\Sigma}$  is diffeomorphic to  $\Sigma$  and is disjoint from  $\partial\Sigma$ . The manifold  $C$  is a collar neighborhood of  $\partial\Sigma$  of the form  $[1/2, 1] \times \partial\Sigma$  where  $\{1\} \times \partial\Sigma = \partial\Sigma$ . Taking a coordinate  $t$  on  $[1/2, 1]$  we can assume that

$$\beta|_C = t \cdot \alpha'$$

for some contact form  $\alpha'$  on  $\partial\Sigma$ . This induces a decomposition of  $[-\epsilon, \epsilon] \times \Sigma$  into two pieces

$$[-\epsilon, \epsilon] \times \Sigma = ([-\epsilon, \epsilon] \times \widehat{\Sigma}) \cup ([-\epsilon, \epsilon] \times \partial\Sigma).$$

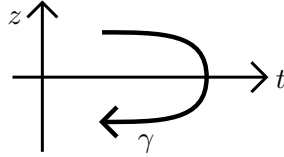


FIGURE 3. The curve  $(z, t) = (z(s), t(s))$  whose image is denoted by  $\gamma$ .

To smooth the corners of  $\partial([-\epsilon, \epsilon] \times \Sigma)$  we can then focus our attention on  $[-\epsilon, \epsilon] \times [1/2, 1] \times \partial\Sigma$ . Let  $(z, t) = (z, t)(s) : [-1, 1] \rightarrow [-\epsilon, \epsilon] \times [1/2, 1]$  be a smooth curve satisfying the following conditions:

- (1)  $(z, t)(-1) = (\epsilon, 1/2)$ ,  $\partial_s(z, t)(-1) = (0, 1)$ , and  $(\partial_s)^k(z, t)(-1) = (0, 0)$  for all  $k > 1$ .
- (2)  $(z, t)(1) = (-\epsilon, 1/2)$ ,  $\partial_s(z, t)(1) = (0, -1)$ , and  $(\partial_s)^k(z, t)(1) = (0, 0)$  for all  $k > 1$ .
- (3)  $(z, t)(-s) = (-z, t)(s)$  for all  $s \in [-1, 1]$ .
- (4) The one form  $zdt - tdz$  evaluated at  $\partial_s(z, t)$  is always positive.

Write  $\gamma$  for the image of the curve  $(z, t)$  in  $[-\epsilon, \epsilon] \times [1/2, 1]$ . See Figure 3.

**Definition 3.4.** In the above notation, let  $\mathcal{N}(\Sigma)$  be the region in  $[-\epsilon, \epsilon] \times \Sigma$  containing  $\{0\} \times \Sigma$  and bounded by

$$(\{-\epsilon, \epsilon\} \times \widehat{\Sigma}) \cup (\gamma \times \partial\Sigma).$$

Here,  $\gamma \times \partial\Sigma$  is considered as a subset of  $[-\epsilon, \epsilon] \times C$ . We call a neighborhood  $\mathcal{N}(\Sigma)$  of  $\Sigma$  constructed in this fashion a standard neighborhood of the Liouville hypersurface  $(\Sigma, \beta) \subset (M, \xi)$ .

**3.3. Contact forms on  $[-\delta, \delta] \times \partial\mathcal{N}(\Sigma)$ .** The hypersurface  $\partial\mathcal{N}(\Sigma)$  is smooth transverse to the vector field  $V_\beta = z\partial_z + X_\beta$ . Now we modify the contact form  $\alpha = dz + \beta$  in a tubular neighborhood of  $\partial\mathcal{N}(\Sigma)$ . By the construction of the neighborhood  $\mathcal{N}(\Sigma)$  of  $\Sigma$  in the previous section,

$$\alpha|_{T(\partial\mathcal{N}(\Sigma))} = \begin{cases} \beta & \text{on } \{-\epsilon, \epsilon\} \times \widehat{\Sigma} \\ \frac{\partial z}{\partial s} ds + t(s) \cdot \alpha' & \text{on } \gamma \times \partial\Sigma. \end{cases}$$

Identify a tubular neighborhood of  $\partial\mathcal{N}(\Sigma)$  with  $[-\delta, \delta] \times \partial\mathcal{N}(\Sigma)$  where  $\partial_\theta = -V_\beta$ . Here  $\theta$  is a coordinate on  $[-\delta, \delta]$ . Then

$$\alpha(\partial_\theta) = -z \quad \text{and} \quad \mathcal{L}_{\partial_\theta} \alpha = -\alpha.$$

Therefore, it follows that

$$\alpha = \begin{cases} e^{-\theta}(\mp\epsilon \cdot d\theta + \beta) & \text{on } [-\delta, \delta] \times \{\pm\epsilon\} \times \widehat{\Sigma} \\ e^{-\theta}(-z_0 \cdot d\theta + \frac{\partial z_0}{\partial s} ds + t_0 \cdot \alpha') & \text{on } [-\delta, \delta] \times \gamma \times \partial\Sigma. \end{cases}$$

The functions  $z_0$  and  $t_0$  in the above equation are the functions  $z(s)$  and  $t(s)$  restricted to  $\partial\mathcal{N}(\Sigma)$ , where  $\partial\mathcal{N}(\Sigma)$  is identified the level set  $\{0\} \times \partial\mathcal{N}(\Sigma) \subset [-\delta, \delta] \times \partial\mathcal{N}(\Sigma)$  of the function  $\theta$ . The contact form  $\alpha$  can then be normalized to obtain a  $\theta$ -invariant contact form  $e^{-\theta}\alpha$ .

We must further perturb the contact form  $e^{-\theta}\alpha$ . Consider the family of contact forms  $\alpha_p$  for  $p \in [0, 1]$  defined by

$$\alpha_p = \begin{cases} e^{-\theta}\alpha & \text{on } [-\delta, \delta] \times \{\pm\epsilon\} \times \widehat{\Sigma} \\ e^{-\theta}\alpha - p \cdot \frac{\partial z_0}{\partial s} ds & \text{on } [-\delta, \delta] \times \gamma \times \partial\Sigma. \end{cases}$$

It is easy to compute that

$$\alpha_p \wedge (d\alpha_p)^n = (e^{-\theta}\alpha) \wedge (d(e^{-\theta}\alpha))^n > 0$$

for all  $p \in [0, 1]$ . Therefore we can apply Moser's trick to the family  $\alpha_p$  to isotop  $[-\delta, \delta] \times \partial(\mathcal{N}(\Sigma))$  in  $M$  so that  $\xi = \text{Ker}(\alpha_1)$  on  $[-\delta, \delta] \times \partial\mathcal{N}(\Sigma)$ . Then we have that  $\widehat{\alpha} := \alpha_1$  is given by

$$(3.3.1) \quad \widehat{\alpha} = \begin{cases} \mp\epsilon \cdot d\theta + \beta & \text{on } [-\delta, \delta] \times \{\pm\epsilon\} \times \widehat{\Sigma} \\ -z_0 \cdot d\theta + t_0 \alpha' & \text{on } [-\delta, \delta] \times \gamma \times \partial\Sigma. \end{cases}$$

Observe that the contact structure determined by the 1-form  $\widehat{\alpha}$  is  $\theta$ -invariant and determines the same contact structure as given by  $\alpha$ . We will soon use  $\widehat{\alpha}$  to give a rigorous definition of the Liouville connect sum. But first, we will review convex hypersurfaces and convex gluing.

**3.4. Convex hypersurfaces.** Throughout this section,  $(M, \xi)$  will be a fixed  $(2n+1)$ -dimensional contact manifold. For simplicity, we only consider closed convex hypersurfaces in this paper.

**Definition 3.5.** A convex hypersurface in  $(M, \xi)$  is a pair  $(S, X)$  consisting of

- (1) a closed, oriented  $(2n)$ -dimensional submanifold  $S \subset M$  and
- (2) a vector field  $X$ , defined on a neighborhood of  $S$ , which is positively transverse to  $S$  and whose flow preserves  $\xi$ .

**Example 3.6.** The pair  $(-\partial\mathcal{N}(\Sigma), \partial_\theta)$  described in the previous section is a convex hypersurface in  $(M, \xi)$ . We may also consider it to be a convex boundary component of  $(M \setminus \text{Int}(\mathcal{N}(\Sigma)), \xi)$ . Note that this choice of orientation on  $\partial\mathcal{N}(\Sigma)$  coincides with the boundary orientation for  $M \setminus \text{Int}(\mathcal{N}(\Sigma))$ , and that  $\partial_\theta$  points transversely out of  $M \setminus \text{Int}(\mathcal{N}(\Sigma))$ .

Let  $(S, X)$  be a convex hypersurface in  $(M, \xi)$ . The vector field  $X$  and contact structure  $\xi$  provide a decomposition of  $S$  into three pieces:

- (1) the *positive region*  $S^+$  consisting of all points in  $S$  for which  $X$  is positively transverse to  $\xi$ ,

- (2) the *dividing set*  $\Gamma_S$  consisting of all points in  $S$  for which  $X \subset \xi$ , and
- (3) the *negative region*  $S^-$  consisting of all points in  $S$  for which  $X$  is negatively transverse to  $\xi$ .

It is easy to see that if we define  $(S', X') = (-S, -X)$ , then  $(S')^+ = -S^-$ ,  $\Gamma_{S'} = \Gamma$ , and  $(S')^- = -S^+$  as oriented manifolds.

Condition (2) of Definition 3.5 is equivalent to saying that for each contact form  $\alpha$  for  $\xi$  we have  $\mathcal{L}_X \alpha = G\alpha$  for some smooth function  $G$  defined in a tubular neighborhood of  $S$ . Now suppose that we identify a neighborhood of  $S$  with  $N(S) := [-1, 1] \times S$ , and  $X = \partial_\theta$  where  $\theta$  is a coordinate on  $[-1, 1]$ . Then we can write  $\alpha = f \cdot d\theta + \beta$  for some function  $f \in C^\infty(N(S), \mathbb{R})$  and  $\beta \in C^\infty([-1, 1], \Omega^1(S))$ . The following proposition allows us to normalize the Lie derivative of  $\alpha$  with respect to the vector field  $\partial_\theta$  on  $N(S)$ .

**Proposition 3.7.** *In the above notation, let  $H \in C^\infty(N(S), \mathbb{R})$  be a smooth function defined in a tubular neighborhood  $N(S)$  of a convex hypersurface  $(S, \partial_\theta)$ . Then we can choose a contact form  $\alpha$  for  $(M, \xi)$  for which  $\mathcal{L}_{\partial_\theta}(\alpha) = H\alpha$  on a neighborhood of  $S$ .*

*Proof.* Let  $\alpha'$  be a contact form on  $N(S)$  satisfying  $\mathcal{L}_{\partial_\theta} \alpha' = G\alpha'$ . We will find a function  $F$  so that  $\alpha = e^F \alpha'$  is as desired. We have that

$$\mathcal{L}_{\partial_\theta}(e^F \alpha') = e^F \left( \frac{\partial F}{\partial \theta} \alpha' + \mathcal{L}_{\partial_\theta} \alpha' \right) = e^F \left( \frac{\partial F}{\partial \theta} + G \right) \alpha'.$$

Therefore, we can find the function  $F$  solving the equation  $H = \frac{\partial F}{\partial \theta} + G$  by defining

$$F(z, x) = \int_0^\theta (H(t, x) - G(t, x)) dt$$

for  $\theta \in [-1, 1]$  and  $x \in S$ . □

Taking the function  $H$  in Proposition 3.7 to be zero, we are guaranteed the existence of a contact form  $\alpha$  for  $(M, \xi)$  such that  $\alpha|_{N(S)} = f \cdot d\theta + \beta$  where  $\frac{\partial f}{\partial \theta} = 0$  and  $\mathcal{L}_{\partial_\theta} \beta = 0$ . With respect to this  $\theta$ -invariant contact form, the contact condition  $(\alpha \wedge (d\alpha)^n > 0)$  is

$$\alpha \wedge (d\alpha)^n = d\theta \wedge (fd\beta + \beta \wedge d_S f) \wedge (d\beta)^{n-1} > 0$$

with respect to the orientation on  $M$ . By analyzing this equation, we are led to the following proposition, the 3-dimensional case of which was first observed in [Gi91], while the general case was described in [CGHH10, §2.2].

**Proposition 3.8.** *In the above notation,  $\Gamma_S$  is a closed, non-empty, codimension-1 submanifold of  $S$ . When oriented as the boundary of  $S^+$ , it is a codimension-2 contact submanifold of  $(M, \xi)$ . The positive region  $S^+$  oriented by  $X$  (and negative region  $S^-$  oriented by  $-X$ ) of  $(S, X)$  is non-empty.*

By further modifying the contact form in a tubular neighborhood of the hypersurface  $S$ , the positive and negative regions of  $S$  (oriented appropriately, with collar neighborhoods of their boundaries removed) can be seen to inherit symplectic forms from the ambient contact manifold as in Definition 1.3. See [CGHH10, §2.2].

**3.5. Convex gluing.** Now suppose that  $(M, \xi)$  and  $(M', \xi')$  are two contact manifolds with convex boundary, where the contact vector fields  $X$  and  $X'$  defined on collar neighborhoods of  $\partial M$  and  $\partial M'$  point out of  $M$  and  $M'$ , respectively. Under certain conditions we can identify the convex boundary components of  $(M, \xi)$  and  $(M', \xi')$  to obtain a larger contact manifold by *convex gluing*.

**Example 3.9.** Let  $X$  and  $X'$  be two smooth, compact  $(n+1)$ -dimensional manifolds with non-empty boundaries which are identified via some orientation reversing diffeomorphism  $\Phi : \partial X \rightarrow \partial X'$ . Consider the associated contact manifolds  $(M, \xi) = (S^*X, \xi_{can})$  and  $(M', \xi') = (S^*X', \xi_{can})$ . Suppose that  $\partial X$  and  $\partial X'$  each have collar neighborhoods  $[-1, 1] \times \partial X$  and  $[-1, 1] \times \partial X'$  on which we identify  $\partial X = \{1\} \times \partial X$



and  $\partial X' = \{1\} \times \partial X'$ . Assume that we have fixed Riemannian metrics on  $X$  and  $X'$  which restrict to product metrics on each collar neighborhood and are such that the map  $\Phi$  is an isometry with respect to the induced metrics on  $\partial X$  and  $\partial X'$ .

Let  $t$  be a coordinate on  $[-1, 1]$ . Then  $\partial_t$  lifts to a vector field on  $S^*X$  which points out of the boundary of  $S^*X$ . Indeed, we can define

$$\tilde{\partial}_t = \partial_s|_{s=0}(\exp(-s \cdot \partial_t))^*.$$

By our choice of metrics, the vector field  $\tilde{\partial}_t$  is tangent to  $S^*X$ , and when restricted to  $S^*M$ , is convex where it is defined. The dividing set may be identified with  $S^*(\partial X)$ , and the positive and negative regions can each be shown to be diffeomorphic to a tubular neighborhood of the zero-section of the cotangent bundle of  $\partial M$ . Define a similar convex vector field on a neighborhood of  $\partial S^*X'$  which points out of the boundary of  $M'$ .

By the restrictions imposed upon metrics used to define  $S^*X$  and  $S^*X'$ , the mapping  $\Phi$  provides a diffeomorphism  $\hat{\Phi} : \partial(S^*X) \rightarrow \partial(S^*X')$ . Then we can use  $\hat{\Phi}$  to perform a convex gluing using the map  $(t, x) \mapsto (-t, \Phi(x))$  from  $[-1, 1] \times \partial M$  to  $[-1, 1] \times \partial M'$ . Under this identification

$$(M, \xi) \cup (M', \xi') = (S^*(X \cup_{\Phi} X'), \xi_{can}).$$

Now we show that the map  $\Upsilon$  from Equation 1.4 determines a convex gluing. Again consider two disjoint Liouville embeddings  $i_1, i_2 : (\Sigma, \beta) \rightarrow (M, \xi)$ . The construction of Section 3.2 above provides two disjoint neighborhoods  $\mathcal{N}(i_1(\Sigma))$  and  $\mathcal{N}(i_2(\Sigma))$  of  $i_1(\Sigma)$  and  $i_2(\Sigma)$ , respectively. The construction in that section also provides us with collar neighborhoods  $[-2\delta, \delta] \times \partial\mathcal{N}(i_1(\Sigma))$  and  $[-2\delta, \delta] \times \partial\mathcal{N}(i_2(\Sigma))$  and a contact form  $\hat{\alpha}$  which is, in each of the collar neighborhoods,  $\theta$ -invariant. Then by the conditions defining the curve  $\gamma$  (used to smooth the corners of  $[-\epsilon, \epsilon] \times \Sigma$ ) and the explicit formula for  $\hat{\alpha}$  in Equation 3.3.1, the map

$$\begin{aligned} \hat{\Upsilon} : [-\delta, \delta] \times \partial\mathcal{N}(i_1(\Sigma)) &\rightarrow [-\delta, \delta] \times \partial\mathcal{N}(i_2(\Sigma)), \quad \hat{\Upsilon}(\theta, x) = (-\theta, \Upsilon(x)) \\ \text{satisfies } \hat{\Upsilon}^* \hat{\alpha}|_{[-\delta, \delta] \times \partial\mathcal{N}(i_2(\Sigma))} &= \hat{\alpha}|_{[-\delta, \delta] \times \partial\mathcal{N}(i_1(\Sigma))}. \end{aligned}$$

Hence, this map can be used to perform the desired convex gluing which defines the Liouville connect sum as described in Section 1.3.

**3.6. Modifications of  $\hat{\Upsilon}$ .** Now we discuss how the map  $\hat{\Upsilon}$  defined in the previous section can be modified. This will be used for constructing contact manifolds in Section 8.

Suppose, as in the previous section, that we have a contact manifold  $(M, \xi)$ , disjoint collar neighborhoods  $[-2\delta, \delta] \times \partial\mathcal{N}(i_1(\Sigma))$  and  $[-2\delta, \delta] \times \partial\mathcal{N}(i_2(\Sigma))$  of two convex boundary components of  $M$ , and a fixed identification  $\hat{\Upsilon} : [-\delta, \delta] \times \partial\mathcal{N}(i_1(\Sigma)) \rightarrow [-\delta, \delta] \times \mathcal{N}(i_2(\Sigma))$ . For notational simplicity, we write

$$S_1 = \partial\mathcal{N}(i_1(\Sigma)) \quad \text{and} \quad S_2 = \mathcal{N}(i_2(\Sigma))$$

for the remainder of this section. Similarly, write  $S$  for an additional copy of  $\partial\mathcal{N}(\Sigma)$ .

Let  $\Phi$  and  $\Psi$  be symplectomorphisms of  $(\Sigma, d\beta)$  which coincide with the identity mapping on a collar neighborhood of  $\partial\Sigma$ . Consider again the decomposition  $S = \{-\epsilon \times \hat{\Sigma}\} \cup \{\epsilon \times \hat{\Sigma}\} \times \{\gamma \times \partial\Sigma\}$  as described in Section 3.1. Define a diffeomorphism  $(\Phi, \Psi) : \partial S_1 \rightarrow S_2$  by

$$(\Phi, \Psi)|_{\{-\epsilon\} \times \hat{\Sigma}} = \Phi, \quad (\Phi, \Psi)|_{\{\epsilon\} \times \hat{\Sigma}}, \quad \text{and} \quad (\Phi, \Psi)|_{\gamma \times \partial\Sigma} = \text{id}_{\gamma \times \partial\Sigma}.$$

We would like to glue together the collar neighborhoods of  $\partial M$  using the map  $\hat{\Upsilon} \circ (\Phi, \Psi) : S_1 \rightarrow S_2$ . However, this map is such that the pullback of  $\hat{\alpha}|_{[-\delta, \delta] \times S_2}$  agrees with  $\hat{\alpha}|_{[-\delta, \delta] \times S_1}$  so as to determine a contact structure on  $M$  after gluing.

By Lemma 3.2 there are isotopies  $\phi_t, \psi_t : [0, 1] \rightarrow \text{Symp}((\hat{\Sigma}, d\beta), \partial\hat{\Sigma})$  for which  $(\Phi \circ \phi_1)^* \beta = \beta + df$  and  $(\Psi \circ \psi_1)^* \beta = \beta + dg$ . Here  $f$  and  $g$  are smooth functions on  $\hat{\Sigma}$  whose supports are disjoint from a collar neighborhood of  $\partial\hat{\Sigma}$ . Provided these isotopies, Lemma 3.3 allows us to find a path  $\hat{\alpha}_t$  of contact forms on  $[-2\delta, \delta] \times \partial\mathcal{N}(i_1(\Sigma))$  such that

- (1)  $\widehat{\alpha}_t$  restricts to a Liouville 1-form on  $\{\theta\} \times \{\pm\epsilon\} \times \widehat{\Sigma}$  for all  $t \in [0, 1]$ ,  $\theta \in [-2\delta, \delta]$ ,
- (2)  $\widehat{\alpha}_0 = \widehat{\alpha}$ ,
- (3)  $\widehat{\alpha}_t|_{[-2\delta, -\frac{3}{2}\delta] \times S_1} = \widehat{\alpha}|_{[-2\delta, -\frac{3}{2}\delta] \times \partial S_1}$  for all  $t \in [0, 1]$ , and
- (4)  $\widehat{\alpha}_1|_{[-\delta, \delta] \times S_1} = (\widehat{\Upsilon} \circ (\Phi \circ \phi_1, \Psi \circ \psi_1))^* \widehat{\alpha}|_{[-\delta, \delta] \times S_2}$ .

Therefore we can use the map

$$(3.6.1) \quad \widehat{\Upsilon} \circ (\Phi \circ \phi_1, \Psi \circ \psi_1) : [-\delta, \delta] \times S_1 \rightarrow [-\delta, \delta] \times S_2$$

to perform a convex gluing. See Figure 4.

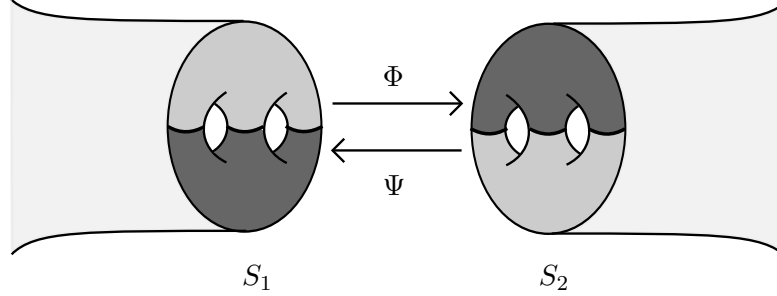


FIGURE 4. This figure gives a schematic description of a convex gluing performed using convex gluing instructions  $(\Phi, \Psi)$  without the correction isotopies  $\phi_t$  and  $\psi_t$ . Our convention is such that the maps  $\Phi$  and  $\Psi$  each send the positive region of one convex boundary component of  $(M, \xi)$  to the negative region of the other convex boundary component.

**Definition 3.10.** We say that the contact manifold obtained from  $(M, \xi)$  by the above construction is described by convex gluing instructions  $(\Phi, \Psi)$ .

**Proposition 3.11.** The above notion is well defined in the sense that the convex gluing construction is independent of the choices  $\phi_t, \psi_t$ , and  $\widehat{\alpha}_t$ . Moreover, the manifold obtained from  $(M, \xi)$  by the convex gluing instructions  $(\Phi, \Psi)$  depends only on the isotopy classes of  $\Phi$  and  $\Psi$  in  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$ .

*Proof.* Suppose that for fixed convex gluing instructions  $(\Phi, \Psi)$  we are given two pairs of symplectic isotopies  $(\phi_t, \psi_t)$  and  $(\phi'_t, \psi'_t)$  for which there are functions  $f, g, f', g'$  as in the above discussion. We would like to show that the contact manifolds determined by the mappings

$$\begin{aligned} \widehat{\Upsilon} \circ (\Phi \circ \phi_1, \Psi \circ \psi_1) : [-\delta, \delta] \times S_1 &\rightarrow [-\delta, \delta] \times S_2 \quad \text{and} \\ \widehat{\Upsilon} \circ (\Phi \circ \phi'_1, \Psi \circ \psi'_1) : [-\delta, \delta] \times S_1 &\rightarrow [-\delta, \delta] \times S_2 \end{aligned}$$

are diffeomorphic. We will find such a diffeomorphism in three steps.

First, we find a diffeomorphism  $\Theta : [-2\delta, \delta] \times S_1 \rightarrow [-2\delta, \delta] \times S_1$  which restricts to the identity mapping on  $\{-2\delta\} \times S_1$  and such that the following diagram commutes:

$$(3.6.2) \quad \begin{array}{ccccc} [-2\delta, \delta] \times S_1 & \xleftarrow{\quad} & [-\delta, \delta] \times S_1 & \xrightarrow{\widehat{\Upsilon} \circ (\Phi \circ \phi_1, \Psi \circ \psi_1)} & [-\delta, \delta] \times S_2 \\ \uparrow \Theta & & \uparrow & & \uparrow \text{id} \\ [-2\delta, \delta] \times S_1 & \xleftarrow{\quad} & [-\delta, \delta] \times S_1 & \xrightarrow{\widehat{\Upsilon} \circ (\Phi \circ \phi_2, \Psi \circ \psi_2)} & [-\delta, \delta] \times S_2 \end{array}$$

This will establish a diffeomorphism between the two manifolds corresponding to the different choices of pairs of correction isotopies  $(\phi_t, \psi_t)$  and  $(\phi'_t, \psi'_t)$ . Write  $\widehat{\alpha}_t$  and  $\widehat{\alpha}'_t$  for paths of contact forms on  $[-2\delta, \delta] \times S_1$

corresponding to the different choices of pairs of correction isotopies. Second, we compare the contact forms  $\Theta^*\hat{\alpha}_1$  and  $\hat{\alpha}_2$  on  $[-2\delta, \delta] \times S_1$ . Finally, we establish the existence of a path of contact forms  $\alpha_t$  interpolating between  $\Theta^*\hat{\alpha}_1$  and  $\hat{\alpha}'_1$  for which  $\alpha_t$  is constant (in variable  $t$ ) near  $\{-2\delta, \delta\} \times S_1$ . Then the first statement of proposition will then follow by an application Gray's stability theorem. The second statement is then a consequence of the independence of choice of  $\phi_t$  and  $\psi_t$ .

**Step 1.** The middle vertical arrow in Equation 3.6.2 is equal to a map which is equal to the identity along  $\gamma \times \partial\Sigma$  and is given by

$$(\theta, -\epsilon, x) \mapsto (\theta, -\epsilon, \phi_1^{-1} \circ \phi'_1(x)), \quad (\theta, \epsilon, x) \mapsto (\theta, \epsilon, \psi_1^{-1} \circ \psi_1(x))$$

along the set  $[-\delta, \delta] \times \{\pm\epsilon\} \times \widehat{\Sigma}$ . Therefore we can find a map  $\Theta$  as desired by defining it as follows: Let  $h : [-2\delta, \delta] \rightarrow [0, 1]$  be a smooth non-decreasing function for which  $h|_{[-2\delta, -\frac{3}{2}\delta]} = 0$  and  $h|_{[-\delta, \delta]} = 1$ . Define  $\Theta$  piecewise by setting it to be equal to

- (1) the identity along  $[-2\delta, \delta] \times (\gamma \times \partial\Sigma)$ ,
- (2)  $(\theta, -\epsilon, x) \mapsto (\theta, -\epsilon, \phi_{h(\theta)}^{-1} \circ \phi'_{h(\theta)}(x))$  on  $[-2\delta, \delta] \times \{-\epsilon\} \times \widehat{\Sigma}$ , and
- (3)  $(\theta, \epsilon, x) \mapsto (\theta, \epsilon, \psi_{h(\theta)}^{-1} \circ \psi'_{h(\theta)}(x))$  on  $[-2\delta, \delta] \times \{\epsilon\} \times \widehat{\Sigma}$ .

As the isotopies  $\phi_t, \psi_t, \phi'_t, \psi'_t$  are all assumed to be equal to the identity mapping on a collar neighborhood of  $\partial\widehat{\Sigma}$ , it follows that the map  $\Theta$  above is indeed smooth. By the properties used to describe the function  $h$ ,  $\Theta$  fits into Equation 3.6.2 so that the diagram commutes and so that it equal to the identity mapping near  $\{-2\delta\} \times S_1$  as desired.

**Step 2.** Now we seek to compute  $\Theta^*\hat{\alpha}_1$  and compare it to  $\hat{\alpha}'_1$ . For each  $\theta \in [-\delta, \delta]$ , the 1-form  $(\widehat{\Upsilon} \circ (\Phi \circ \phi_1, \Psi \circ \psi_1))^*\hat{\alpha}$  is equal to

- (1)  $\beta + df$  when restricted to  $T(\{\theta\} \times \{-\epsilon\} \times \widehat{\Sigma})$  and
- (2)  $\beta + dg$  when restricted to  $T(\{\theta\} \times \{\epsilon\} \times \widehat{\Sigma})$ .

Here  $\hat{\alpha}$  is considered as a 1-form on  $[-\delta, \delta] \times S_2$ . Similarly, we have that  $(\widehat{\Upsilon} \circ (\Phi \circ \phi'_1, \Psi \circ \psi'_1))^*\hat{\alpha}$  is equal to

- (1)  $\beta + df'$  when restricted to  $T(\{\theta\} \times \{-\epsilon\} \times \widehat{\Sigma})$  and
- (2)  $\beta + dg'$  when restricted to  $T(\{\theta\} \times \{\epsilon\} \times \widehat{\Sigma})$ .

By the above equations and the construction described in Lemma 3.3 we can assume that the contact forms  $\hat{\alpha}_1$  and  $\hat{\alpha}'_1$  are such that

- (1)  $\hat{\alpha}_t = \hat{\alpha}'_t = \pm d\theta + \beta$  along the sets  $[-2\delta, -\frac{3}{2}\delta] \times \{\mp\epsilon\} \times \widehat{\Sigma}$  for all  $t \in [0, 1]$ ,
- (2)  $\hat{\alpha}_1 = d\theta + df + \beta$  on  $[-\delta, \delta] \times \{-\epsilon\} \times \widehat{\Sigma}$ ,
- (3)  $\hat{\alpha}_1 = -d\theta + dg + \beta$  on  $[-\delta, \delta] \times \{\epsilon\} \times \widehat{\Sigma}$ ,
- (4)  $\hat{\alpha}'_1 = d\theta + df' + \beta$  on  $[-\delta, \delta] \times \{-\epsilon\} \times \widehat{\Sigma}$ , and
- (5)  $\hat{\alpha}'_1 = -d\theta + dg' + \beta$  on  $[-\delta, \delta] \times \{\epsilon\} \times \widehat{\Sigma}$ .

Hence it follows from the construction of the map  $\Theta$  that  $\Theta^*\hat{\alpha}_1$  coincides with  $\hat{\alpha}'_1$  along the sets  $[-2\delta, -\frac{3}{2}\delta] \times S_1$  and  $[-\delta, \delta] \times S_1$ . Moreover, these contact forms are such that the restriction of either to the tangent space of each  $\{\theta\} \times \{\pm\epsilon\} \times \widehat{\Sigma}$  is Liouville.

**Step 3.** To finish the proof we seek to show that the contact structures determined by the contact forms  $\Theta^*\hat{\alpha}_1$  and  $\hat{\alpha}'_1$  are equivalent. Let  $\alpha_t$  be the path of contact forms on  $[-2\delta, \delta] \times S_1$  given by

$$\alpha_t = \begin{cases} \Theta^*\hat{\alpha}_{1-h(t)} & : t \in [0, 1] \\ \hat{\alpha}'_{h(t-1)} & : t \in [1, 2] \end{cases}$$

which smoothly interpolates between  $\Theta^*\hat{\alpha}_1$  and  $\hat{\alpha}'_1$ . By the construction of  $\Theta$  and Lemma 3.3 we can extend this family of contact forms to  $[-2\delta, \delta] \times S_2$  by setting

$$\alpha_t|_{[-\delta, \delta]} = ((\hat{\Upsilon} \circ (\Phi \circ \phi'_1, \Psi \circ \psi'_1)^{-1})^* \alpha_t|_{[-\delta, \delta] \times S_1}$$

and requiring that  $\alpha_t - \alpha$  vanishes along  $[-2\delta, -\frac{3}{2}\delta] \times S_2$ . Gray's stability theorem applied to the path of contact forms  $\alpha_t$  then provides us with a 1-parameter family of diffeomorphisms interpolating between the contact structures determined by  $\alpha_0 = \Theta^*\hat{\alpha}_1$  and  $\alpha_2 = \hat{\alpha}'_1$ . The stability theorem applies in this case, even though the ambient manifold

$$[-2\delta, \delta] \times S_1 \cup |_{\hat{\Upsilon} \circ (\Phi \circ \phi'_1, \Psi \circ \psi'_1)} [-2\delta, \delta] \times S_2$$

is not closed, as the  $\alpha_t$  is constant in variable  $t$  on a collar neighborhood of its boundary; see the discussion preceding Lemma 3.14 in [MS99, §3.2].  $\square$

#### 4. A NEIGHBORHOOD THEOREM FOR LIOUVILLE SUBMANIFOLDS OF HIGH CODIMENSION

The purpose of this section is to show that every Liouville submanifold  $(\Sigma, \beta)$  of a contact manifold  $(M, \xi)$  whose codimension is greater than one embeds into a Liouville hypersurface  $(\hat{\Sigma}, \hat{\beta}) \subset (M, \xi)$  which smoothly retracts onto  $\Sigma$ .

The proof consists of two parts. The first part of this proof, established in Theorem 4.3, is an existence result asserting that the total space of every symplectic disk bundle over a Liouville domain admits the structure of a Liouville domain in a natural way. It is easy to construct an exact symplectic form on the total space of such a bundle using, for example Thurston's technique [MS99, Theorem 6.3]. That being said, the content of Theorem 4.3 is that the Liouville vector field can be made transverse to the boundary of this disk bundle and so our construction will rely on a different technique. The second part follows the standard Gray-Moser-Weinstein argument used to establish neighborhood theorems in contact and symplectic geometry, showing that we can isotope a submanifold  $\hat{\Sigma}$  of  $(M, \xi)$  containing  $\Sigma$  so that  $\alpha|_{T\hat{\Sigma}}$  coincides with the model Liouville 1-form provided by Theorem 4.3 where  $\alpha$  is a fixed contact form for  $(M, \xi)$ .

Throughout this section, for a path  $\gamma = \gamma(t)$  in a manifold we will abbreviate the tangent vector  $\partial_t|_{t=0}\gamma(t)$  based at  $\gamma(0)$  by  $\partial_t\gamma(t)$  for the purpose of simplifying equations.

##### 4.1. Some existence results.

**Lemma 4.1.** *Let  $\pi : E \rightarrow M$  be a rank  $2m$  vector bundle over a manifold  $M$  equipped with a smooth section  $\omega$  of  $E^* \wedge E^* \rightarrow X$  which is symplectic on each fiber of  $E$ . Then there is a 1-form  $\lambda \in \Omega^1(E)$  on the total space  $E$  such that*

- (1) *on each fiber  $E_x$ ,  $x \in M$ , there is a coordinate system  $p_j, q_j, j = 1, \dots, m$  for which*

$$\lambda|_{TE_x} = \frac{1}{2} \sum_1^m (p_j dq_j - q_j dp_j),$$

- (2)  *$d\lambda(\partial_t(v_x + tw_x), \partial_t(v_x + tw'_x)) = \omega(w_x, w'_x)$  for all  $v_x, w_x, w'_x \in E_x, x \in M$ ,*  
 (3)  *$\lambda$  and  $d\lambda$  are both annihilated by vectors tangent to the zero section of  $E$ , and*  
 (4)  *$\lambda = \frac{1}{2}d\lambda(R_E, *)$ , where  $R_E$  is the radial vector field on  $E$ .*

**Definition 4.2.** A vector bundle  $E$  equipped with a fiber-wise bilinear form  $\omega$  as described in the statement of the above lemma will be referred to a symplectic vector bundle and will denoted by the pair  $(E, \omega)$ .

*Proof.* Fix a complex structure  $J$  and bundle metric  $\langle *, * \rangle$  on  $E$  so that  $\omega(*, J*) = \langle *, * \rangle$ . Such a complex structure and bundle metric always exist as can be seen in [MS99, §2.6]. Fix a unitary connection  $\nabla$  on  $E$  compatible with  $J$  and  $\omega$ . Such a connection satisfies

$$(4.1.1) \quad J\nabla_X v = \nabla_X(Jv) \quad \text{and} \quad X\langle v, w \rangle = \langle \nabla_X V, W \rangle + \langle V, \nabla_X W \rangle$$

for all pairs of sections  $v, w$  of  $E$  and vector fields  $X$  on  $M$ . The connection  $\nabla$  determines a splitting of  $TE$  into *vertical* and *horizontal* subspaces  $TE = V \oplus H$ . At a vector  $v_x, x \in M$ , the vertical subspace  $V_{v_x}$  is spanned by tangent vectors of the form

$$\partial_t(v_x + tw_x) \quad w_x \in E_x.$$

The horizontal subspace  $H_{v_x}$  of  $T_{v_x}E$  is spanned by vectors of the form  $\partial_t(v_{\gamma(t)})$  for sections  $v_{\gamma(t)}$  over paths  $\gamma(t) : (-\epsilon, \epsilon) \rightarrow M$  satisfying  $v_{\gamma(0)} = v_x$  and  $\nabla_{\partial_t(\gamma(t))} v_{\gamma(t)} = 0$  for all  $t \in (-\epsilon, \epsilon)$ . Denote by  $\pi_V$  and  $\pi_H$  the projections of  $TE$  onto  $V$  and  $H$ , respectively. Define  $\phi : V \rightarrow E$  to be the mapping  $\partial_t(v_x + tw_x) \mapsto x_x$ . Write  $R_E$  for the radial vector field on  $E$ .

We claim that the 1-form

$$(4.1.2) \quad \lambda := \frac{1}{2} \omega(\phi(R_E), *) \circ \phi \circ \pi_V \in \Omega^1(E)$$

is as in the statement of the lemma. To verify this claim we will need to perform some calculations in a local coordinate system.

Let  $B \subset M$  be a ball in  $M$  over which we have a trivialization of  $\pi^{-1}(B)$  by linearly independent sections  $v_j, w_j, j = 1, \dots, m$  satisfying

$$Jv_j = w_j, \quad Jw_j = -v_j, \quad \omega(v_i, v_j) = \omega(w_i, w_j) = 0 \quad \text{and} \quad \omega(v_i, w_j) = \delta_{i,j}.$$

This implies that  $\omega = \sum_1^m v_j^* \wedge w_j^*$  where  $v_j^*, w_j^*$  is the associated dual basis of  $v_j, w_j$ . Such a trivialization exists as a consequence of the fact that every finite dimensional unitary vector space  $(\mathbb{R}^{2n}, J, \langle *, * \rangle)$  is isomorphic to  $\mathbb{C}^n$  equipped with its standard unitary structure. Let  $x_i$  be a system of coordinates on  $B$ . From this coordinate system and the trivialization of  $\pi^{-1}(B)$  we obtain a coordinate system  $x_i, p_j, q_j$  on  $\pi^{-1}(B)$  by associating each  $(x, p, q)$  with the vector  $\sum_1^m (p_j v_j + q_j w_j)$  in  $E$  over the point in  $B$  determined by the  $x_i$ . For notational simplicity we will abbreviate  $\sum_1^m (p_j v_j + q_j w_j)$  as  $\binom{p}{q}$  when viewed as a vector over a point in  $B$ . With respect to these coordinates, we have that at the point  $(x, p, q)$

$$(4.1.3) \quad \begin{aligned} \phi(\partial_{p_j}) &= v_j, & \phi(\partial_{q_j}) &= w_j, & \phi(\partial_{x_i}) &= \nabla_{\partial_{x_i}} \binom{p}{q}, \\ \pi_V(\partial_{p_j}) &= \partial_{p_j}, & \pi_V(\partial_{q_j}) &= \partial_{q_j}, & \pi_H(\partial_{p_j}) &= \pi_H(\partial_{q_j}) = 0, \\ \pi_V(\partial_{x_i}) &= \partial_t(\binom{p}{q} + t\phi(\partial_{x_i})), & \text{and} & & \pi_H(\partial_{x_i}) &= \partial_{x_i} - \pi_V(\partial_{x_i}). \end{aligned}$$

Using these equations together with the observation that we may write the radial vector field on  $E$  locally as  $R_E = \sum p_j \partial_{p_j} + q_j \partial_{q_j}$  so that  $\phi(R_E) = \binom{p}{q}$ , we obtain the expression

$$(4.1.4) \quad 2\lambda = \left( \sum_1^m (p_j dq_j - q_j dp_j) \right) + \left( \sum_1^{\dim(M)} \omega\left(\binom{p}{q}, \nabla_{\partial_{x_i}} \binom{p}{q}\right) dx_i \right).$$

Then we can use Equation 4.1.1 to calculate

$$\begin{aligned}
 (4.1.5) \quad 2d\lambda &= 2\left(\sum_1^m dp_j \wedge dq_j\right) \\
 &+ \left(\sum_{i=1}^m \sum_{j=1}^{\dim(M)} (\omega(v_j, \nabla_{\partial_{x_i}} \binom{p}{q}) + \omega(\binom{p}{q}, \nabla_{\partial_{x_j}} v_i)) dp_i \wedge dx_j\right) \\
 &+ \left(\sum_{i=1}^m \sum_{j=1}^{\dim(M)} (\omega(w_j, \nabla_{\partial_{x_i}} \binom{p}{q}) + \omega(\binom{p}{q}, \nabla_{\partial_{x_j}} w_i)) dq_i \wedge dx_j\right) \\
 &+ \left(\sum_{i,k=1}^{\dim(M)} (\omega(\nabla_{\partial_{x_k}} \binom{p}{q}, \nabla_{\partial_{x_i}} \binom{p}{q}) + \omega(\binom{p}{q}, \nabla_{\partial_{x_k}} \nabla_{\partial_{x_i}} \binom{p}{q})) dx_k \wedge dx_i\right).
 \end{aligned}$$

Now it follows from Equations 4.1.4 and 4.1.5 that the 1-form  $\lambda$  defined in Equation 4.1.2 satisfies properties (1)-(3) listed in the statement of the lemma. Property (4) follows from a straightforward calculation.  $\square$

In what follows we will need to further understand the 2-form  $d\lambda$  on  $E$  determined by Equation 4.1.2. Therefore we provide a coordinate-free expression of Equation 4.1.5. For vector fields  $X$  and  $Y$  on  $E$ , this is given by

$$\begin{aligned}
 (4.1.6) \quad d\lambda(X, Y) &= \omega(\phi \circ \pi_V(X), \phi \circ \pi_V(Y)) \\
 &+ \frac{1}{2} \omega((\nabla_{T\pi(X)} \nabla_{T\pi(Y)} - \nabla_{T\pi(Y)} \nabla_{T\pi(X)} + \nabla_{T\pi([Y, X])}) \circ \phi(R_E), \phi(R_E)).
 \end{aligned}$$

A calculation showing that this expression coincides with the one given in Equation 4.1.5 is not difficult provided the equations in the proof above, and so is omitted.

**Theorem 4.3.** *A sufficiently small neighborhood  $\mathbb{D}$  of the zero section of a rank  $2m$  symplectic vector bundle  $\pi : (E, \omega) \rightarrow \Sigma$  over a Liouville domain  $(\Sigma, \beta)$  naturally carries the structure of a Liouville domain  $(\widehat{\Sigma}, \widehat{\beta})$ , where  $\widehat{\Sigma}$  is obtained by rounding the corners  $\partial(\pi^{-1}(\partial\Sigma))$  of  $\mathbb{D}$ . The 1-form  $\widehat{\beta}$  is such that*

- (1) *on each fiber  $\mathbb{D}_x$ ,  $x \in \Sigma$ , there is a coordinate system  $p_j, q_j, j = 1, \dots, m$  for which*

$$\widehat{\beta}|_{T\mathbb{D}_x} = \frac{1}{2} \sum_1^m (p_j dq_j - q_j dp_j),$$

- (2)  *$d\lambda(\partial_t(v_x + tw_x), \partial_t(v_x + tw'_x)) = \omega(w_x, w'_x)$  for all  $v_x, w_x, w'_x \in \mathbb{D}_x, x \in X$ , and*  
 (3)  *$\widehat{\beta}$  coincides with  $\beta$  when restricted to the tangent space of the zero-section of  $\mathbb{D}$ .*

The proof of this theorem is a continuation of the construction described in the proof of Lemma 4.1 and makes use of the unitary structure  $(\omega, J, \langle *, * \rangle)$  and connection  $\nabla$  on the vector bundle  $E$  described there. Our use of the word “naturally” in the statement of this theorem indicates that the Liouville domain  $(\widehat{\Sigma}, \widehat{\beta})$  - considered modulo homotopy of Liouville 1-forms - is independent of  $J, \langle *, * \rangle$ , and  $\nabla$ . Indeed, the space of complex structures compatible with  $(E, \omega)$  is contractible and the space of unitary connections on a fixed unitary vector bundle  $(E, \omega, J, \langle *, * \rangle)$  is an affine vector space. Hence the following proof will indicate that a smooth 1-parameter family of such choices  $(J_t, \langle *, * \rangle_t, \nabla_t)$  will give rise to a smooth 1-parameter family  $\widehat{\beta}_t$  of Liouville 1-forms, possibly after shrinking the size of the tubular neighborhood  $\mathbb{D}$  of the zero-section of  $E \rightarrow \Sigma$ .

*Proof.* We define  $\widehat{\beta} = \lambda + \pi^* \beta$ . By the properties of  $\lambda$  listed in the statement of Lemma 4.1 it follows that on a sufficiently small neighborhood  $\mathbb{D}_\epsilon := \{v \in E : \langle v, v \rangle \leq \epsilon\}$  of the zero-section of  $E$  the 2-form  $d\widehat{\beta}$  is

symplectic. As  $\pi^*\beta$  and  $d\pi^*\beta = \pi^*d\beta$  are annihilated by tangent vectors in the vertical subspaces of  $TE$ ,  $\hat{\beta}$  satisfies the properties listed in the statement of the theorem. Therefore, all that remains to be shown is that the vector field  $X_{\hat{\beta}}$  determined by the equation  $d\hat{\beta}(X_{\hat{\beta}}, *) = \hat{\beta}$  is positively transverse to the boundary of  $\mathbb{D}_\epsilon$  for a sufficiently small constant  $\epsilon > 0$ .

Let  $X_\beta$  be the Liouville vector field of  $(\Sigma, \beta)$ . The connection  $\nabla$  on  $E$  provides a means of lifting  $X_\beta$  to a horizontal vector field  $\widetilde{X}_\beta$  on  $E$ . This vector field is uniquely determined by the properties that  $\pi_V(\widetilde{X}_\beta) = 0$  and  $T\pi(\widetilde{X}_\beta|_{v_x}) = X_\beta|_x$  for each  $x \in \Sigma$  and  $v_x \in \pi^{-1}(x)$ . These properties indicate that  $d\pi^*\beta(\widetilde{X}_\beta, *) = \pi^*\beta(*)$ .

Write  $X_{\hat{\beta}} = \frac{1}{2}R_E + \widetilde{X}_\beta + Z$  for the Liouville vector field associated to  $\hat{\beta}$  on  $E$ . We will now analyze the correction term  $Z$ . From Lemma 4.1 and the definition of  $\widetilde{X}_\beta$  we calculate

$$\hat{\beta} = d\hat{\beta}(X_{\hat{\beta}}, *) = \hat{\beta} + d\lambda(\widetilde{X}_\beta, *) + d\hat{\beta}(Z, *).$$

Equivalently, we have the equations

$$(4.1.7) \quad -d\lambda(\widetilde{X}_\beta, *) = d\hat{\beta}(Z, *) = d\lambda(Z, *) + d\pi^*\beta(Z, *).$$

As  $d\hat{\beta}$  is symplectic, the above equations uniquely determine  $Z$ . Equation 4.1.6 indicates that  $d\lambda(\widetilde{X}_\beta, *)$  evaluates to zero on vertical vector fields. This, the fact that  $d\pi^*\beta(Z, *)$  evaluates to zero on vertical tangent vectors for any  $Z$ , and Equation 4.1.7 tell us that  $d\lambda(Z, *)$  must also vanish along the vertical distribution. By the fact that  $d\lambda$  is non-degenerate when restricted to  $V \subset TE$ , we conclude that the vector field  $Z$  must be contained in the horizontal distribution  $H \subset TE$ .

Denote by  $\rho : E \rightarrow \mathbb{R}$  the “length-squared” function  $\rho(v_x) = \langle v_x, v_x \rangle$ . As the connection  $\nabla$  is unitary, vector fields  $v_{\gamma(t)}$  along curves  $\gamma(t) : (-\epsilon, \epsilon) \rightarrow$  for which  $\nabla_{\partial_t \gamma(t)} v_{\gamma(t)} = 0$  satisfy  $\rho(v_{\gamma(t)}) = \rho(v_{\gamma(0)})$  for all  $t \in (-\epsilon, \epsilon)$ . Hence,  $\rho$  is constant along integral curves of horizontal vector fields. Now, using the fact that both  $\widetilde{X}_\beta$  and  $Z$  are horizontal we calculate that at a point  $v_x \in E$  the vector field  $X_{\hat{\beta}}$  satisfies

$$d\rho(X_{\hat{\beta}}) = d\rho(\frac{1}{2}R_E) + d\rho(\widetilde{X}_\beta) + d\rho(Z) = \frac{1}{2}\partial_t(\langle (1+t)v_x, (1+t)v_x \rangle) + 0 + 0 = \rho(v_x).$$

From this calculation we conclude that for any  $\epsilon > 0$  the vector field  $X_{\hat{\beta}}$  is positively transverse to the boundary of  $\mathbb{D}_\epsilon$  along the boundary of  $\pi^{-1}(\text{Int}(\Sigma))$ .

To finish our proof we will show that for  $\epsilon$  sufficiently small,  $X_{\hat{\beta}}$  is transverse to the boundary of  $\mathbb{D}_\epsilon$  along  $\pi^{-1}(\partial\Sigma)$ . To see this, observe that for any  $\epsilon > 0$

- (1)  $\frac{1}{2}R_E$  is tangent to the boundary of  $\mathbb{D}_\epsilon$  along  $\pi^{-1}(\partial\Sigma)$ ,
- (2) the vector field  $\widetilde{X}_\beta$  is positively transverse to the boundary of  $\mathbb{D}_\epsilon$  along  $\pi^{-1}(\partial\Sigma)$ , and
- (3) the vector field  $Z$  vanishes along the zero-section of  $\mathbb{D}_\epsilon$  as can be seen by Equations 4.1.6 and 4.1.7.

Therefore we conclude that for  $\epsilon$  sufficiently small,  $X_{\hat{\beta}}$  is transverse to  $\partial\mathbb{D}_\epsilon$  along  $\pi^{-1}(\partial\Sigma)$ . After having fixed such an  $\epsilon$ , we can round the corners of  $\mathbb{D}_\epsilon$  to obtain a manifold with smooth boundary  $\widehat{\Sigma} \subset \mathbb{D}_\epsilon$  for which  $X_{\hat{\beta}}$  is positively transverse to  $\partial\widehat{\Sigma}$  so that  $\hat{\beta}$  is a Liouville 1-form on  $\widehat{\Sigma}$ .  $\square$

The same construction can be easily applied to symplectic vector bundles over contact manifolds.

**Theorem 4.4.** *Let  $(M, \xi)$  be a contact manifold with contact form  $\alpha$ . Then a sufficiently small neighborhood  $\mathbb{D}$  of the zero section of a rank  $2m$  symplectic vector bundle  $\pi : (E, \omega) \rightarrow M$  naturally carries the structure of a contact manifold  $(\mathbb{D}, \xi_E)$ . The contact structure  $\xi_E$  can be described as  $\text{Ker}(\hat{\alpha})$  for a 1-form  $\hat{\alpha}$  such that*

(1) on each fiber  $\mathbb{D}_x$ ,  $x \in M$ , there is a coordinate system  $p_j, q_j, j = 1, \dots, m$  for which

$$\hat{\alpha}|_{T\mathbb{D}_x} = \frac{1}{2} \sum_1^m (p_j dq_j - q_j dp_j),$$

(2)  $d\hat{\alpha}(\partial_t(v_x + tw_x), \partial_t(v_x + tw'_x)) = \omega(w_x, w'_x)$  for all  $v_x, w_x, w'_x \in \mathbb{D}_x$ ,  $x \in X$ , and  
 (3)  $\hat{\alpha}$  coincides with  $\alpha$  when restricted to the tangent space of the zero-section of  $\mathbb{D}$ .

To prove this theorem, we may define  $\hat{\alpha} = \lambda + \pi^*\alpha$ . The 1-form  $\hat{\alpha}$  is contact on a sufficiently small tubular neighborhood of the zero-section of  $E$  by Equation 4.1.6. It is well known that the contact structure on a tubular neighborhood of a contact submanifold is uniquely determined by its symplectic normal bundle. However, the author is unsure as to whether or not the a proof of the existence of a contact structure on the total space of a symplectic disk bundle over a given contact manifold has been written anywhere.

#### 4.2. Liouville submanifolds of codimension $> 1$ .

**Theorem 4.5.** *Let  $(M, \xi)$  be a  $(2n+1)$ -dimensional contact manifold with contact form  $\alpha$  and suppose that  $\Sigma$  is a compact  $(2k)$ -dimensional submanifold of the interior of  $M$  such that  $k < n$  and  $\alpha|_{T\Sigma}$  is a Liouville 1-form on  $\Sigma$ . Then  $\Sigma$  admits a neighborhood of the form  $[-\epsilon, \epsilon] \times \hat{\Sigma}$  on which  $\alpha = dz + \hat{\beta}$  for a Liouville 1-form  $\hat{\beta}$  on  $\hat{\Sigma}$ . The manifold  $\hat{\Sigma}$  is obtained by rounding the corners  $\pi^{-1}(\partial\Sigma)$  of a disk bundle  $\pi : \mathbb{D} \rightarrow \Sigma$ . Moreover, there is a coordinate system  $(p_j, q_j)$  on each fiber of the disk bundle on which*

$$\alpha|_{\pi^{-1}(x)} = \frac{1}{2} \sum_1^{n-k} (p_j dq_j - q_j dp_j).$$

*Proof.* We write  $\beta = \alpha|_{T\Sigma} \in \Omega^1(\Sigma)$ . The fact that  $d\alpha|_{T\Sigma} = d\beta$  is symplectic is equivalent to the fact that the Reeb vector field  $R_\alpha$  for  $\alpha$  is nowhere tangent to  $\Sigma$ . Thus we may decompose the vector bundle  $\xi|_\Sigma$  into a direct sum  $\xi|_\Sigma = \xi_\Sigma \oplus \xi_\Sigma^\perp$  where

$$\xi_\Sigma = \{v - \alpha(v)R_\alpha : v \in T\Sigma\} \quad \text{and} \quad \xi_\Sigma^\perp = \{v \in \xi_\Sigma : d\alpha(v, *)|_{\xi_\Sigma} = 0\}.$$

Observe that  $\xi_\Sigma$  is isomorphic to  $T\Sigma$  and that  $d\alpha$  is fiber-wise symplectic on each of  $\xi_\Sigma$  and  $\xi_\Sigma^\perp$ . This follows from the computation

$$d\alpha(v - \alpha(v)R_\alpha, w - \alpha(w)R_\alpha) = d\beta(v, w)$$

for each pair of vectors  $v, w \in T\Sigma$ , coupled with the fact that  $\xi_\Sigma^\perp$  is by definition the symplectic complement of  $\xi_\Sigma$  with respect to  $d\alpha$ .

Fix a Riemannian metric  $\langle *, * \rangle$  on  $M$  and denote by  $\exp : TM \rightarrow M$  the associated exponential map, sending each tangent vector  $v_x \in T_x M$  to  $\gamma(\|v_x\|)$  where  $\gamma(t)$  is the unique geodesic in  $M$  satisfying  $\gamma(0) = x$  and  $\partial_t \gamma(0) = v_x$ . To be completely rigorous, we should either restrict the domain of  $\exp$  or assume that the metric  $\langle *, * \rangle$  is complete so that the mapping is defined. However, this will not be an issue as we will be applying  $\exp$  to vectors of arbitrarily small length along  $TM|_\Sigma$  and have assumed that  $\Sigma$  is contained in the interior of  $M$ .

Denote by  $\mathbb{D}_\epsilon$  the collection of vectors in  $\xi_\Sigma^\perp$  of length less than or equal to  $\epsilon$  for an arbitrarily small constant  $\epsilon > 0$ . As  $\exp$  has the property that for each  $v_x \in TM$ ,  $\partial_t(\exp(tv_x)) = v_x$ , that we can choose the constant  $\epsilon$  to be small enough so that  $\exp(\mathbb{D}_\epsilon)$  is embedded and symplectic with respect to the 2-form  $d\alpha$ . For simplicity, we shall henceforth use the symbol  $\mathbb{D}_\epsilon$  to denote the image  $\exp(\mathbb{D}_\epsilon) \subset M$ . These assumptions guarantee that  $R_\alpha$  is transverse to  $\mathbb{D}_\epsilon$ . By appealing to this transversality and using the assumption that  $\epsilon$  is chosen to be small enough so that  $\mathbb{D}_\epsilon$  is contained in the interior of  $M$ , we obtain a neighborhood  $[-\delta, \delta] \times \mathbb{D}_\epsilon$  of  $\mathbb{D}_\epsilon$  via the mapping

$$(z, x) \mapsto \exp(z \cdot R_\alpha)(x) \quad \text{for} \quad x \in \mathbb{D}_\epsilon.$$



Indeed, choosing the constant  $\delta > 0$  to be sufficiently small, we may assume that this mapping is an embedding. Note that  $\alpha|_{[-\delta, \delta] \times \mathbb{D}_\epsilon} = dz + \alpha|_{T\mathbb{D}_\epsilon}$ , where we consider  $\alpha|_{T\mathbb{D}_\epsilon}$  to be a  $z$ -invariant 1-form which evaluates to zero on  $\partial_z$ .

Possibly after further shrinking  $\epsilon$  there is an isotopy  $\Phi_t : \mathbb{D}_\epsilon \rightarrow \mathbb{D}_\epsilon$ ,  $t \in [0, 1]$ , such that  $\Phi_0 = \text{id}_{\mathbb{D}_\epsilon}$ ,  $\Phi_t|_\Sigma = \text{id}_\Sigma$  for all  $t$ , and  $d\alpha|_{T\mathbb{D}_\epsilon}$  is equal to the symplectic 2-form  $\Phi_1^* d\hat{\beta}$  determined by the symplectic vector bundle  $(\xi_\Sigma^\perp, d\alpha|_{\xi_\Sigma^\perp})$  as described in Theorem 4.3. This is a consequence of the fact that the symplectic forms  $d\alpha|_{T\mathbb{D}_\epsilon}$  and  $\hat{\beta}$  agree on  $T\mathbb{D}|_\Sigma$ . See [MS99, Lemma 3.14]. Therefore  $\alpha - \Phi_1^* \hat{\beta}$  is closed. We also observe that  $\hat{\beta}$  and  $\alpha$  agree on  $T\mathbb{D}_\epsilon|_\Sigma$ , implying that  $\alpha|_{T\mathbb{D}_\epsilon} - \Phi_1^* \hat{\beta}$  is exact. This allows us to find a function  $f \in C^\infty(\mathbb{D}_\epsilon, \mathbb{R})$  satisfying  $\alpha|_{T\mathbb{D}_\epsilon} - \Phi_1^* \hat{\beta} = df$  and  $f|_\Sigma = 0$ .

If necessary, further shrink  $\epsilon$  so that  $|f| < \delta$  on  $\mathbb{D}_\epsilon$ . To complete the proof, isotop  $\mathbb{D}_\epsilon$  to the graph of the function  $-f$  in  $[-\delta, \delta] \times \mathbb{D}_\epsilon$ . Then  $\alpha|_{T\mathbb{D}_\epsilon} = \hat{\beta}$ . By the properties of  $\hat{\beta}$  listed in Theorem 4.3, we see that after rounding the corners  $\partial(\pi^{-1}(\partial\Sigma))$  of  $\mathbb{D}_\epsilon$  we obtain a Liouville hypersurface  $(\hat{\Sigma}, \hat{\beta}) \subset (M, \xi)$  as desired.  $\square$

## 5. EXAMPLES OF LIOUVILLE HYPERSURFACES

In this section we give some simple examples of Liouville hypersurfaces in contact manifolds.

**5.1. Legendrian graphs.** Let  $L \subset (M, \xi)$  be a Legendrian submanifold of the  $(2n+1)$ -dimensional contact manifold  $(M, \xi)$ . Then  $L$  admits a tubular neighborhood

$$N(L) = [-\epsilon, \epsilon] \times \mathbb{D}^*L \quad \text{where} \quad \xi|_{N(L)} = \text{Ker}(dz - \lambda_{\text{can}})$$

where  $z$  is a coordinate on  $[-\epsilon, \epsilon]$ . Then  $\{0\} \times \mathbb{D}^*L$  is a Liouville hypersurface in  $(M, \xi)$ . More generally, we can construct interesting Liouville hypersurfaces by considering *Legendrian graphs* in  $(M, \xi)$ .

**Definition 5.1.** Let  $(M, \xi)$  be a  $(2n+1)$ -dimensional contact manifold.

- (1) A Legendrian graph in  $(M, \xi)$  is a pair  $(L, \phi)$  where  $L$  is a compact  $(n)$ -manifold and a Legendrian immersion  $\phi : L \rightarrow M$  with only double point singularities, possibly occurring along  $\partial L$ . At each double point  $x = \phi(p) = \phi(q)$ ,  $p \neq q$ , we require that  $\xi_x = (T_p L) \oplus (T_q L)$ .
- (2) A ribbon of a Legendrian graph  $(L, \phi)$  is a smooth, compact  $(2n)$ -dimensional submanifold  $R(\phi(L))$  of  $M$  such that
  - (a)  $\phi(L) \subset R(\phi(L))$ ,
  - (b)  $R(\phi(L))$  deformation retracts onto  $\phi(L)$ , and
  - (c) for  $x \in R(\phi(L))$ ,  $\xi_x = T_x R(\phi(L))$  if and only if  $x \in \phi(L) \subset R(\phi(L))$ .

If  $(L, \phi)$  is a Legendrian graph in  $(M, \xi)$  then  $\phi(L)$  admits a neighborhood of the form

$$N(\phi(L)) = [\epsilon, \epsilon] \times R(\phi(L)) \quad \text{such that} \quad \xi|_{N(\phi(L))} = \text{Ker}(dz + \beta)$$

where  $\beta$  is a Liouville 1-form on the ribbon  $R(\phi(L))$ . A ribbon of a Legendrian graph  $(L, \phi)$  is Liouville diffeomorphic to a plumbing of the cotangent bundle of  $L$  at the double points of the immersion  $\phi$ . For examples of Liouville domains constructed from plumbings of cotangent bundles, see [EI91, §7]. The above definition can easily be extended to *isotropic graphs*.

**5.2. Inclusions.** By the results of Section 4, the Liouville hypersurface property is well behaved with respect to inclusion mappings. Suppose that  $(\Sigma, \beta)$  is a Liouville hypersurface in a contact manifold  $(C, \zeta)$  of dimension  $(2k+1)$  and we realize  $(C, \zeta)$  as a contact submanifold of a  $(2n+1)$ -dimensional contact manifold  $(M, \xi)$ . Then  $(\Sigma, \beta)$  is a Liouville submanifold of  $(M, \xi)$  and we can apply Theorem 4.5.

**5.3. Liouville hypersurfaces in unit cotangent bundles.** Let  $X$  be a smooth manifold and denote its unit cotangent bundle equipped with its canonical contact structure by  $(M, \xi)$ . Denote the projection map  $M = S^*X \rightarrow X$  by  $\pi$ .

Suppose that  $X$  is oriented and that  $Y$  is an oriented, codimension-one submanifold of  $X$ . As noted in Example 3.9,  $S = \pi^{-1}(Y)$  is a convex hypersurface in  $(M, \xi)$  whose dividing set is  $\Gamma = S^*Y$ . The orientation on  $Y$  allows us to specify one of the components of  $\pi^{-1}(Y) \setminus \Gamma$  as the positive region  $S^+$ , by taking the vector field  $\partial_t$  from Example 3.9 to be oriented so that it is positively transverse to  $Y$ . In this situation  $(S^+, -\lambda_{can})$  is Liouville-homotopic to the cotangent disk bundle  $(\mathbb{D}^*Y, -\lambda_{can})$  of  $Y$ .

Here is another method of finding Liouville hypersurfaces in cotangent bundles: Suppose that  $(\Sigma, \beta)$  is a Liouville domain of dimension less than  $\dim(X)$  and that  $\Phi : \Sigma \rightarrow X$  is an immersion. Let  $\text{Ann}(\Phi)$  be the sub-bundle of  $\Phi^*T^*X$  consisting of the vectors which annihilate  $T\Sigma$  via the natural pairing of tangent and cotangent vectors. Possibly after multiplying  $\beta$  by a small positive constant, we may assume that the length  $\beta$  at each point with respect to the pullback metric on  $T^*X$  - induced by the Riemannian metric on  $TX$  - inherited from  $X$  is less than 1. Let  $s$  be a section of  $\text{Ann}(\Phi)$  such that  $\|s + \beta\| = 1$  with respect to the metric on  $T^*X$ . Now lift the image  $\Phi(\Sigma)$  of the immersion  $\Phi$  in  $X$  to  $M$  via the section  $s + \beta$ . As the dimension of  $\Sigma$  is less than the dimension of  $X$ , this determines a Liouville embedding of  $(\Sigma, \beta)$  into  $(M, \xi)$ , possibly after a  $C^\infty$ -small perturbation. By the results of Section 4, there is a disk bundle in  $M$  whose zero-section is  $\Sigma \subset M$ , which is a Liouville hypersurface of  $(M, \xi)$ .

**5.4. Counterexamples: Cabling and overtwisted submanifolds.** Now we give examples of contact embeddings which do not bound Liouville hypersurfaces. Let  $(M, \xi)$  be a 5-dimensional contact manifold and let  $T^2 \subset M$  be a Legendrian torus. Identify  $S^*T^2 = T^3 \subset M$  as the boundary of the ribbon of  $T^2$ . After fixing a trivialization of the normal bundle of  $T^3$ , we can identify a tubular neighborhood  $N$  of  $T^3$  with

$$N = T^3 \times \mathbb{D}^2, \quad \xi|_N = \text{Ker}(\sin(2\pi z)dx + \cos(2\pi z)dy + r^2 d\theta)$$

where we consider coordinates  $(x, y, z)$  on  $T^3 = ([0, 1]/\sim)^3$ ,  $0 \sim 1$  and polar coordinates on  $\mathbb{D}^2$ .

For  $R \in (0, 1)$ , consider the map  $\Psi_{n,R} : T^3 \rightarrow M$  given by

$$\Psi_{n,R}(x, y, z) = ((x, y, nz), Re^{2\pi iz}) \in N = T^3 \times \mathbb{D}^2.$$

$$\text{Then } \Psi_{n,R}^* \alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy + 2\pi R^2 dz$$

where  $\alpha$  is the contact form on  $N$  used above to describe  $\xi|_N$ . The map  $\Psi_{n,R}$  is a contact embedding whose image is null-homologous. By applying Moser's trick to the family of contact forms  $\Psi_{n,R}^* \alpha$  as  $R$  goes to zero, it can be seen that the contact structure on  $T^3$  determined by  $\Psi_{n,R}$  for any  $R \in (0, 1)$  is the well known  $(T^3, \xi_n)$ , where

$$\xi_n = \text{Ker}(\sin(2\pi nz)dx + \cos(2\pi nz)dy).$$

According to [El96], for  $n > 1$  these 3-tori are not symplectically fillable and so cannot bound a Liouville hypersurface in  $(M, \xi)$ . However, as noted in [Gi94], these tori are weakly symplectically fillable.

This example can be generalized to find “cables” of arbitrary codimension-2 submanifolds in a manifold of any dimension. Suppose that  $C$  is a closed, codimension-2 submanifold of a manifold  $M$  whose normal bundle is trivial. Fix an identification of a tubular neighborhood  $N(C)$  of  $C$  with  $\mathbb{D}^2 \times C$ .

Suppose that we have a non-trivial representation  $\rho : \pi_1(C) \rightarrow \mathbb{Z}/q\mathbb{Z}$  of the fundamental group of  $C$  into a finite cyclic group. Denote by  $\pi : \tilde{C}_\rho \rightarrow C$  the associated cover with deck transformation group  $\rho(\pi_1(C)) \subset \mathbb{Z}/q\mathbb{Z}$ . Define  $E_\rho \rightarrow C$  to be the complex line bundle over  $C$  determined by

$$E_\rho = (\mathbb{C} \times \tilde{C}_\rho) / \sim \quad \text{where} \quad (x, v) \sim (\gamma \cdot x, e^{\frac{\rho(\gamma)}{q} 2\pi i} \cdot v) \quad \forall \gamma \in \rho(\pi_1(C)).$$

Here  $\gamma \cdot x$  denotes the action of  $\gamma \in \rho(\pi_1(C))$  on  $x \in \tilde{C}_\rho$  by deck transformation. Let  $D_\rho$  be the unit disk bundle in  $E_\rho$  and fix a trivialization  $\Phi : D_\rho \rightarrow \mathbb{D}^2 \times C$ . Such a trivialization exists as  $E_\rho$  has a flat connection with  $S^1 = U(1)$  holonomy. This gives rise to an embedding of  $\tilde{C}_\rho$  into  $M$  by

- (1) identifying  $\tilde{C}_\rho$  with the section  $\{1\} \times \tilde{C}_\rho \subset \mathbb{C} \times \tilde{C}_\rho$ ,
- (2) embedding  $\{1\} \times \tilde{C}_\rho$  into  $D_\rho$  using the projection  $\mathbb{C} \times \tilde{C}_\rho \rightarrow E_\rho$ , and finally
- (3) applying the embedding  $\Phi$  into  $M$  using the identification  $\mathbb{D}^2 \times C = N(C)$ .

In the simplest case, of an oriented unknot  $C$  in  $S^3$  with the obvious surjective representation  $\rho : \pi_1(S^1) \rightarrow \mathbb{Z}/q\mathbb{Z}$ , this construction yields a  $(p, q)$ -torus knot. The integer  $p$  is determined by the trivialization of  $N(C)$ .

As in the case of  $T^3$  above, we can see that if  $C$  is a contact submanifold  $(C, \zeta)$  of a contact manifold  $(M, \xi)$  this embedding is a contact embedding of  $(\tilde{C}_\rho, \pi^*\zeta)$  into  $(M, \xi)$ . The fundamental class of  $\tilde{C}_\rho$  in  $H_{2n-1}(M, \mathbb{Z})$  is  $|\rho(\pi_1(C))| \cdot [T]$ .

This procedure can be used to find embeddings of overtwisted contact 3-manifolds into contact 5-manifolds. For example if  $(C, \zeta)$  is exactly symplectically fillable with filling  $(\Sigma, \beta)$ , by taking a Liouville embedding of  $(\Sigma, \beta)$  into  $(M, \xi)$ , we have a null-homologous contact embedding of  $(C, \zeta)$  into  $(M, \xi)$  as the boundary of  $(\Sigma, \beta) \subset (M, \xi)$ . If  $(C, \zeta)$  has a finite-cyclic, overtwisted cover then apply the above construction.

Examples of closed, Stein fillable contact 3-manifolds which have finite-cyclic, overtwisted covers can be found in the work of Gompf [Gom98, Proposition 5.1] and Honda [Hon00a, Proposition 5.1]. It would be interesting to have examples of high ( $> 3$ ) dimensional contact manifolds which are symplectically fillable (in any of the senses of Section 2.2) and which admit finite covers which are not symplectically fillable. To the author's knowledge, there are no such known examples in the literature.

The cabling construction described above can be used in other contexts to produce interesting codimension-2 submanifolds of a given manifold. For example, let  $\Sigma$  be a closed, connected, 2-dimensional symplectic submanifold of a 4-dimensional symplectic manifold  $(W, \omega)$ . Assume that  $\Sigma$  has genus  $g(\Sigma) \geq 1$  and self-intersection number  $[\Sigma] \cdot [\Sigma] = 0$ . This implies that the symplectic normal bundle to  $\Sigma$  is trivial so that we can identify a neighborhood of  $\Sigma$  in  $W$  with  $N(\Sigma) = \mathbb{D}^2 \times \Sigma$  where  $\omega|_{N(\Sigma)} = d\lambda_{std} + \sigma$  for some symplectic form  $\sigma$  on  $\Sigma$ . Applying the cabling construction to a surjective representation  $\rho : \pi_1(\Sigma) \rightarrow \mathbb{Z}/q\mathbb{Z}$  produces a closed, connected, embedded, symplectic surface  $\tilde{\Sigma}_\rho \subset W$  with genus  $g(\tilde{\Sigma}_\rho) = q(g - 1) + 1$  and fundamental homology class  $[\tilde{\Sigma}_\rho] = q \cdot [\Sigma] \in H_2(W, \mathbb{Z})$ .

## 6. MORE BASIC CONSEQUENCES OF DEFINITION 1.3

In this section we outline more of “the basics” of Liouville hypersurfaces in contact manifolds. Special results for contact 3-manifolds are listed in Section 6.2. We begin with a discussion concerning a special family of contact vector fields called *contact dilations*.

**6.1. Contact dilations.** Every standard neighborhood  $\mathcal{N}(\Sigma)$  of a Liouville hypersurface  $(\Sigma, \beta)$  in a contact manifold  $(M, \xi)$  admits a special contact vector field which points out of  $\partial\mathcal{N}(\Sigma)$ . Let  $X_\beta$  be the Liouville vector field for  $(\Sigma, \beta)$ . Then the vector field  $V_\beta = z\partial_z + X_\beta$  satisfies

$$(6.1.1) \quad \mathcal{L}_{V_\beta} \alpha = \alpha,$$

and points transversely out of  $\mathcal{N}(\Sigma)$  along its boundary.

**Definition 6.1.** A vector field for which there exists a contact form  $\alpha$  such that the Lie derivative condition of Equation 6.1.1 is satisfied will be referred to as a contact dilation.

Note that if a compact contact manifold  $(M, \xi)$  admits a contact dilation which is defined on all of  $M$ , then the boundary of  $M$  is necessarily non-empty.

**Remark 6.2.** We will see in Section 7.1 that the existence of a contact dilation in a neighborhood of Liouville hypersurface is essential to the construction of the symplectic cobordism  $(W, \lambda)$  described in the statement of Theorem 1.9.

In the above notation,  $(S = \partial\mathcal{N}(\Sigma), V_\beta)$  is then a convex surface in  $(M, \xi)$ .

**Proposition 6.3.** A hypersurface  $\Sigma \subset M$  is Liouville if and only if there is a convex hypersurface  $(S, X)$  in  $(M, \xi)$  for which  $\Sigma$  is the complement of a collar neighborhood of  $\partial S^+$  in  $S^+$ .

*Proof.* The “if” statement is a consequence of our ability to normalize contact forms in tubular neighborhoods in convex hypersurfaces as mentioned in Section 3.4. For the “only if” statement, let  $\Sigma \subset M$  be a Liouville hypersurface. Then, in the notation of Section 3.1,  $\Sigma$  is isotopic through a family of Liouville hypersurfaces to  $\{\epsilon\} \times \widehat{\Sigma} \subset (\partial\mathcal{N}(\Sigma))^+$  for a standard neighborhood  $\mathcal{N}(\Sigma)$  of  $\Sigma$ .  $\square$

Liouville hypersurfaces provide a simple means of partially characterizing contact dilations.

**Proposition 6.4.** Let  $(M, \xi)$  be a compact  $(2n + 1)$ -dimensional contact manifold with contact 1-form  $\alpha$ . Suppose that  $(M, \xi)$  equipped with a vector field  $V$  which points out of  $\partial M$  transversely and is a contact dilation for the contact form  $\alpha$ . Then  $(M, \xi)$  is contact-diffeomorphic to a standard neighborhood of a Liouville hypersurface.

*Proof.* Let  $z : M \rightarrow \mathbb{R}$  be the function  $z = \alpha(V)$  and let  $R_\alpha$  be the Reeb field for  $\alpha$ . Then

$$(6.1.2) \quad \begin{aligned} \alpha &= \mathcal{L}_V \alpha = d\alpha(V, *) + d(\alpha(V)) = d\alpha(V, *) + dz, \quad \text{and} \\ 1 &= \alpha(R_\alpha) = d\alpha(V, R_\alpha) + dz(R_\alpha) = dz(R_\alpha). \end{aligned}$$

Therefore  $dz$  is never zero and the vector field  $R_\alpha$  is transverse to every level set of the function  $z$ .

As  $V$  is positively transverse to  $\partial M$ ,  $(M, \xi)$  has convex boundary. By the definition of the function  $z$ , the associated dividing set on  $\partial M$  is

$$\Gamma_{\partial M} = \partial M \cap \{z = 0\}.$$

Moreover, by the preceding paragraph  $\Sigma := \{z = 0\}$  is a Liouville hypersurface in  $M$  with boundary equal to  $\Gamma_{\partial M}$ . As  $\Sigma$  is transverse to  $\partial M$  in  $M$ ,  $\Sigma$  admits a tubular neighborhood  $[-\epsilon, \epsilon] \times \Sigma$  on which  $\alpha = dz + \beta$  by the results in Section 3.1. Here  $\beta = \alpha|_{T\Sigma}$ . Indeed, the transversality of  $R_\alpha$  with  $\Sigma$  implies that  $d\alpha$  is symplectic on  $\Sigma$ . Moreover,  $V$  is tangent to  $\Sigma$ , and when considered as a vector field on  $\Sigma$  is the Liouville vector field for  $(\Sigma, \beta)$ .

Again, from the definition of  $z$  and Equation 6.1.2,  $V$  points transversely out of  $\mathcal{N}(\Sigma)$  and is non-vanishing on  $M \setminus \mathcal{N}(\Sigma)$ . Therefore we can identify  $M \setminus \mathcal{N}(\Sigma)$  as being contained in  $[0, \infty) \times \partial\mathcal{N}(\Sigma)$  by placing a coordinate  $s$  on  $[0, \infty)$  such that  $\partial_s = V$ . Because of the transversality of  $V$  with  $\partial M$ , we have that  $\partial M$  is the graph of a function  $\partial\mathcal{N}(\Sigma) \rightarrow [0, \infty)$  contained in  $[0, \infty) \times \partial\mathcal{N}(\Sigma)$ .  $\square$

The above proposition is false without the assumption that the vector field  $V$  is positively transverse to  $\partial M$ . For example, if  $S$  is a closed convex hypersurface in a contact manifold  $(M, \xi)$ , then Proposition 3.7 indicates that  $S$  admits a tubular neighborhood of the form  $N(S) = [-1, 1] \times S$  on which there is a contact form  $\alpha$  admitting a contact dilation pointing into  $N(S)$  along  $\{-1\} \times S$  and out of  $N(S)$  along  $\{1\} \times S$ .

**6.2. Special properties when  $\text{Dim}(M) = 3$ .** In this section we state some results specific to Liouville surfaces in 3-dimensional contact manifolds. We assume that the reader is familiar with the basics of transverse knots, self-linking numbers, and characteristic foliations. For more information, see [Et05].

**Proposition 6.5.** Suppose that  $(M, \xi)$  is a 3-dimensional contact manifold.

- (1)  $\Sigma \subset M$  is a Liouville surface if and only if it is isotopic through a 1-parameter family of Liouville surfaces to a ribbon (Definition 5.1) of some Legendrian graph.

- (2)  $(M, \xi)$  is tight if and only if every one of its Liouville surfaces is genus minimizing among all embedded surfaces with the same boundary and in the same boundary-relative homology class.
- (3)  $(M, \xi)$  is tight if and only if the boundary of every Liouville surface in  $(M, \xi)$  is transversely non-destabilizeable.

Item (2) is taken from [BCV09, Theorem 8]. Additional results regarding ribbons of Legendrian graphs in overtwisted contact manifolds can also be found in [BCV09]. In the second item we use the generalized definition of Seifert genus for links. A surface  $\Sigma$  bounding a link  $L$  is of *minimal genus* if it realizes the maximal Euler characteristic among all surfaces bounding  $L$  with no sphere components. Note that (2) can be used to compute genera of certain topological knots and links as in [Ga81, Theorem 2].

The necessity of taking into account relative homology classes in item (2) can be seen in the following simple example: Let  $\Sigma^-$  be a torus with a disk removed and let  $\Sigma^+$  be a genus 2 surface with a disk removed. Denote by  $\Sigma^+ \cup \Sigma^-$  the closed genus three surface obtained by identifying the boundary components of  $\Sigma^+$  and  $\Sigma^-$ . Let  $(M, \xi)$  be the  $S^1$ -invariant contact structure on  $S^1 \times (\Sigma^+ \cup \Sigma^-)$  whose (oriented) dividing set on each slice  $\{\theta\} \times (\Sigma^+ \cup \Sigma^-)$  is  $\partial\Sigma^+$ . Then each  $\{\theta\} \times \Sigma^+$  is a Liouville surface which does not realize the Seifert genus of its boundary.  $(M, \xi)$  is universally tight by Giroux's criterion and so is tight.

*Proof.* (1) For the first item, we see that  $\Sigma$  is isotopic (as in the statement above) to  $(\partial\mathcal{N}(\Sigma))^+$ . Using the Legendrian realization principle [Hon00a, §3.3.1] it is easy to construct a Legendrian graph onto which  $(\partial\mathcal{N}(\Sigma))^+$  deformation retracts.

(2) This is immediate from (1) and [BCV09, Theorem 8].

(3) Suppose that  $(M, \xi)$  is overtwisted. Then the “standard” transverse unknot given by the boundary of a Liouville disk  $(\mathbb{D}^2, \lambda_{std}) \subset (M, \xi)$  is transversely destabilizeable. See, for example, [Et05, Theorem 3.2].

Now suppose that  $(M, \xi)$  is tight and that  $\Sigma$  is a Liouville surface in  $(M, \xi)$  bounding a transverse link  $T = \sqcup T_j$ . Then the *Thurston-Bennequin inequality* applies. This asserts that for any transverse link  $T$  bounding an embedded, oriented surface  $\Sigma \subset M$ , the inequality

$$(6.2.1) \quad s\ell(T, \Sigma) \leq -\chi(\Sigma)$$

is satisfied. Eliashberg's proof [El92] of this inequality relies on the inequality

$$(6.2.2) \quad s\ell(T, \Sigma) + \chi(\Sigma) = e_-^\Sigma - h_-^\Sigma \leq 0$$

where  $e_-^\Sigma$  and  $h_-^\Sigma$  are the number of negative elliptic and negative hyperbolic singularities of a generic characteristic foliation on the surface  $\Sigma$ , respectively. As the Liouville condition is an open condition, we may assume that the characteristic foliation of  $\Sigma$  is generic – i.e. its singularities are isolated and of Morse type – and apply Equation 6.2.2. In this case the numbers  $e_-^\Sigma$  and  $h_-^\Sigma$  are both zero as by definition there is a contact form  $\alpha$  for  $(M, \xi)$  for which  $d\alpha|_{T\Sigma}$  is symplectic. Hence,  $\Sigma$  satisfies the equality  $s\ell(T, \Sigma) = -\chi(\Sigma)$ .

Now suppose that some component  $T_j$  of  $T$  is a transverse stabilization of a transverse knot  $T'_j$  in the complement of  $M \setminus (T \setminus T_j)$ . Then we would be able to find another embedded surface  $S$ , which is smoothly isotopic to  $\Sigma$  bounding  $T' = (T \setminus T_j) \cup T'_j$ . Therefore, we would have

$$s\ell(T', S) = s\ell(T, \Sigma) + 1 = -\chi(\Sigma) + 1$$

contradicting Equation 6.2.1. □

It would be interesting to know what transverse knots bound Liouville surfaces.

**Question 6.6.** *Is it possible to characterize the transverse links in  $(S^3, \xi_{std})$  which bound Liouville surfaces in terms of braid theory?*

Implicit in the above question is the fact, due to Bennequin [Be82], that every transverse link in  $(S^3, \xi_{std})$  can be represented as a transverse braid. See [Et05, §2.4] and the references therein.

It would also be interesting to know whether or not a Liouville surface bounding a given transverse link is unique.

**Question 6.7.** *Does there exist a transverse link  $T$  in  $(S^3, \xi_{std})$  and two Liouville surfaces  $\Sigma, \Sigma' \subset (S^3, \xi_{std})$  for which  $\partial\Sigma = \partial\Sigma' = T$  and such that  $\Sigma$  is not isotopic to  $\Sigma'$  through a family of Liouville surfaces?*

An answer to the above question in the affirmative would be analogous to the non-uniqueness of minimal genus Seifert surfaces of topological knots in  $\mathbb{R}^3$ . See, for example, [R90, §5.A].

## 7. SYMPLECTIC COBORDISMS ASSOCIATED TO LIOUVILLE CONNECT SUMS

This section is devoted to the proof of Theorem 1.9. In Section 7.1 we prove part the first part of the theorem, establishing the existence of the cobordism  $(W, \lambda)$ . Then, in Section 7.2, we prove the second statement of Theorem 1.9 by showing that when the Liouville hypersurface  $(\Sigma, \beta)$  is Stein, the cobordism  $(W, \lambda)$  admits a Weinstein handle decomposition. Contact manifolds appearing as the convex boundaries of weak symplectic cobordisms are dealt with in Section 7.3.

### 7.1. Symplectic handle attachment.

*Proof of Theorem 1.9.* We construct a handle  $H_\Sigma$ , from a standard neighborhood (Section 3.1)  $\mathcal{N}(\Sigma)$  of a Liouville hypersurface  $(\Sigma, \beta) \subset (M, \xi)$ , and then attach  $H_\Sigma$  to convex boundary of the compact symplectization  $([\frac{1}{2}, 1] \times M, t \cdot \alpha)$  of  $(M, \xi)$ .

**Step 1: Setup.** Let  $(M, \xi)$  be a  $(2n + 1)$ -dimensional contact manifold with a fixed contact form  $\alpha$  and let  $(\Sigma, \beta)$  be a  $2n$ -dimensional Liouville domain. Let  $i_1$  and  $i_2$  be embeddings of  $\Sigma$  into  $M$  whose images are disjoint and satisfy  $i_1^* \alpha = i_2^* \alpha = \beta$ . By the results of Section 3.1 there exists tubular neighborhoods  $\mathcal{N}(i_j(\Sigma))$  of  $i_j(\Sigma)$ ,  $j = 1, 2$ , with smooth convex boundary.

We fix an additional copy of a standard neighborhood  $\mathcal{N}(\Sigma)$  from which  $H_\Sigma$  will be constructed. The contact form  $\alpha$  is assumed to take the form  $\alpha = dz + \beta$  in each of these neighborhoods.

**Step 2: Model geometry on the handle.** Consider the symplectic manifold

$$(H_\Sigma, \omega_\beta) = ([-1, 1] \times \mathcal{N}(\Sigma), d\theta \wedge dz + d\beta),$$

where  $\theta$  is the coordinate on  $[-1, 1]$ . Consider the vector field  $V_\beta = z\partial_z + X_\beta$  on  $\mathcal{N}(\Sigma)$  satisfying  $\mathcal{L}_{V_\beta} \alpha = \alpha$ , where  $X_\beta$  is the Liouville vector field for  $(\Sigma, \beta)$ , as described in Section 6.1. Then viewing  $V_\beta$  as a  $\theta$ -invariant vector field on  $H_\Sigma$ ,  $\omega_\beta(V_\beta, *) = -zd\theta + \beta$  and  $\mathcal{L}_{V_\beta} \omega_\beta = \omega_\beta$ . It follows that  $V_\beta$  is a symplectic dilation of  $(H_\Sigma, \omega_\beta)$ . This vector field points transversely out of  $\partial H_\Sigma$  along  $[-1, 1] \times \partial\mathcal{N}(\Sigma)$  and is tangent to  $\{\pm 1\} \times \mathcal{N}(\Sigma)$ . Therefore,  $-zd\theta + \beta$  is a  $\theta$ -invariant contact 1-form on  $[-1, 1] \times \partial\mathcal{N}(\Sigma)$  inducing the  $\theta$ -invariant contact structure on  $[-1, 1] \times \partial\mathcal{N}(\Sigma)$  described in Section 3.3.

In order to be able to attach  $\{\pm 1\} \times \mathcal{N}(\Sigma)$  to the “top”  $\{1\} \times M$  of the compact symplectization of  $(M, \xi)$  in a way so that  $t \cdot \alpha$  on  $[1/2, 1] \times M$  extends over  $H_\Sigma$ , we must modify  $V_\beta$  so that the  $\{\pm 1\} \times \mathcal{N}(\Sigma)$  is concave in the sense of Definition 2.4.

To achieve concavity, let  $Y$  be the Hamiltonian vector field for the function  $z\theta$  on  $H_\Sigma$  with respect to the symplectic form  $\omega_\beta$ . Then  $Z := V_\beta - Y$  is a symplectic dilation of  $\omega_\beta$ . Moreover,  $Z$  points into  $H_\Sigma$  along  $\{\pm 1\} \times \mathcal{N}(\Sigma)$  and out of  $H_\Sigma$  along  $[-1, 1] \times \partial\mathcal{N}(\Sigma)$ . Define

$$(7.1.1) \quad \lambda = \omega_\beta(Z, *) = -\theta dz - 2zd\theta + \beta.$$

Then  $d\lambda = \omega_\beta$  and  $\lambda|_{\{\pm 1\} \times \mathcal{N}(\Sigma)} = \mp dz + \beta$ .

**Step 3: Gluing the handle.** Now we attach  $H_\Sigma$  to the compact symplectization of  $(M, \xi)$ . This will be achieved by defining embeddings

$$i_- : [-1, 0) \times \mathcal{N}(\Sigma) \rightarrow [1, \infty) \times \mathcal{N}(i_1(\Sigma)) \quad \text{and} \quad i_+ : (0, 1] \times \mathcal{N}(\Sigma) \rightarrow [1, \infty) \times \mathcal{N}(i_2(\Sigma))$$

for which  $i_\pm^*(t \cdot \alpha) = \lambda$ . Here the  $\mathcal{N}(i_j(\Sigma)) \times [1, \infty)$ ,  $j = 1, 2$ , are considered as subsets of  $[1/2, \infty) \times M$ .

By the properties of the vector field  $Z$  on  $[-1, 1] \times \mathcal{N}(\Sigma)$  listed in the previous step, we can consider  $[-1, 0) \times \mathcal{N}(\Sigma)$  as being contained in  $[1, \infty) \times \mathcal{N}(\Sigma)$  by identifying  $\{-1\} \times \mathcal{N}(\Sigma) \subset H_\Sigma$  with  $\{1\} \times \mathcal{N}(\Sigma)$  and the dilating vector field  $Z$  with the vector field  $t\partial_t$ . This defines the embedding  $i_-$ . Define  $i_+$  similarly.

It follows that  $i_\pm^*(t\alpha) = \lambda$  as defined in Equation 7.1.1. Note that  $Ti_\pm(Z) = t\partial_t$ . Therefore,  $\lambda$  defined on  $H_\Sigma$  and  $t\alpha$  can be patched together, determining a globally defined Liouville 1-form on  $W^\square := ([1/2, 1] \times M) \cup_{i_\pm} H_{(\Sigma, \beta)}$ . Similarly,  $Z$  and  $t\partial_t$  can be patched together producing the associated Liouville vector field.

**Step 4: Edge rounding.** Now we must round the edges of  $W^\square$  and show that the resulting smooth manifold  $W$  gives an cobordism from  $(M, \xi)$  to  $\#_{(\Sigma, \beta)}(M, \xi)$ .

The equation

$$Ti_-(\partial_\theta)|_{\theta=-1} = \partial_t - 2z\partial_z + X_\beta$$

implies that  $\{1\} \times (M \setminus N_1) \cup i_-([-1, 0) \times \partial\mathcal{N}(\Sigma))$  is not smooth. To correct this, we can find a function  $f : M \rightarrow (1/2, \infty)$  such that

- (1)  $f = 1$  along  $\partial N(i_1(\Sigma))$  and
- (2) the vector field  $\partial_t - 2z\partial_z + X_\beta$ , defined in an arbitrarily small tubular neighborhood of  $\partial\mathcal{N}(\Sigma)$ , is tangent to the graph of  $f$  where it is defined.

Then the union of  $i_-([-1, 0) \times \partial\mathcal{N}(\Sigma))$  with the graph of  $f$  (over the complement of  $N_1$ ) is a smooth manifold. Further correct the function  $f$  near  $N(i_2(\Sigma))$  so as to round the edges of  $\partial W^\square$  near  $i_+((0, 1] \times \mathcal{N}(\Sigma))$ .

Now, we have a smooth  $(2n + 2)$ -dimensional exact symplectic cobordism  $W$  obtained from smoothing the corners of  $W^\square$  and identifying the 1-forms  $t \cdot \alpha$  and  $\lambda$  via the maps  $i_-$  and  $i_+$ . To finish our proof, we will show that the convex boundary of  $W$  is the contact manifold  $\#_{(\Sigma, \beta)}(M, \xi)$ .

It suffices to check that the induced contact structure on  $[-1, 1] \times \partial\mathcal{N}(\Sigma)$  coincides with the  $\theta$ -invariant contact structure described in Step 2. To see this, observe that the maps  $i_-$  and  $i_+$  defined above identify  $[-1, 0) \times \mathcal{N}(\Sigma)$  and  $(0, 1] \times \mathcal{N}(\Sigma)$  with the graphs of functions over the sets

$$N_j := N(i_j(\Sigma)) \setminus \left( \lim_{s \rightarrow \infty} \exp(-s \cdot V_\beta)(N_j) \right) \quad j = 1, 2$$

in  $[1, \infty) \times M$ . In other words, the  $(N_j, \xi)$  are the same contact manifolds as given by the completion of neighborhoods of the  $\partial N(i_j(\Sigma))$  with respect to the contact vector fields  $-V_\beta$ . As  $\xi|_{N_j'}$  gives the desired contact structure, our construction is complete.

**7.2. Weinstein handle decomposition.** Again, let  $(M, \xi)$  be a  $(2n + 1)$ -dimensional contact manifold with contact form  $\alpha$ . In this section, we again adopt all of the notational conventions of the previous section. Suppose that the Liouville domain  $(\Sigma, \beta)$  is Stein. Then there is a decomposition

$$(7.2.1) \quad \sqcup (\mathbb{D}^{2n}, \lambda_{std}) = (\Sigma_0, \beta_0) \subset \cdots \subset (\Sigma_n, \beta_n) = (\Sigma, \beta)$$

of  $(\Sigma, \beta)$  as described in Theorem 2.11(3). Consider the symplectic cobordism  $(W, \lambda)$  from  $(M, \xi)$  to  $\#_\Sigma(M, \xi)$  described in the previous section. We will use the filtration described in Equation 7.2.1 to filter  $(W, \lambda)$  as

$$([1/2, 1] \times M, t \cdot \alpha) = (W_{-1}, \lambda_{-1}) \subset (W_0, \lambda_0) \subset \cdots \subset (W_n, \lambda_n) = (W, \lambda)$$

where each  $(W_k, \lambda_k)$  is obtained from  $(W_{k-1}, \lambda_{k-1})$  by attaching some number of  $(2n + 2)$ -dimension Weinstein  $(k + 1)$ -handles.

Consider the restrictions of the embeddings  $i_1$  and  $i_2$  to  $(\Sigma_j, \beta_j)$ . Let  $N_{j,1}$  and  $N_{j,2}$  denote neighborhoods of  $i_1(\Sigma_j)$  and  $i_2(\Sigma_j)$  given by

$$[-\epsilon, \epsilon] \times \Sigma_k \subset [-\epsilon, \epsilon] \times \Sigma \quad \text{with} \quad \alpha = dz + \beta_k.$$

Define  $(W_k, \lambda_k)$  to be the cobordism associated to the Liouville connect sum of  $(M, \xi)$  along  $N_{k,1}$  and  $N_{k,2}$ . The filtration of Equation 7.2.1 induces a filtration of handles  $(H_{\Sigma_k}, \lambda_k|_{H_{\Sigma_k}} = -\theta dz - 2zd\theta + \beta_k)$ . Therefore we have

$$(7.2.2) \quad (H_{\Sigma_0}, \lambda_0) \subset \cdots \subset (H_{\Sigma_n}, \lambda_n).$$

As each of the  $(W_k, \lambda_k)$  is given by the attachment of  $(H_{\Sigma_k}, \lambda_k)$  to the compact symplectization of  $(M, \xi)$ , then Equation 7.2.2 induces the filtration

$$([1/2, 1] \times M, t \cdot \alpha) = (W_{-1}, \lambda_{-1}) \subset (W_0, \lambda_0) \subset \cdots \subset (W_n, \lambda_n) = (W, \lambda).$$

Note that  $(W_0, \lambda_0)$  is obtained from  $([1/2, 1] \times (M, \xi), t \cdot \alpha)$  by attachment of some number of  $(2n + 2)$ -dimensional Weinstein 1-handles. This may be seen by comparing the first part of the proof of Theorem 1.9 and the exposition in Section 2.3.

Now consider the cobordism  $(W_{k-1}, \lambda_{k-1})$ . Let  $\Lambda_j$ ,  $j = 1, \dots, m$  be the core  $k$ -disks of the  $(2n)$ -dimensional Weinstein  $k$  handles that are attached to the boundary of  $(\Sigma_{k-1}, \beta_{k-1})$  to obtain  $(\Sigma_k, \beta_k)$ . Consider, for each  $j$ , two copies  $\Lambda_{j,1}$  and  $\Lambda_{j,2}$  of  $\Lambda_j$  living in the boundary of  $(W_{k-1}, \lambda_{k-1})$  as

$$\begin{aligned} \Lambda_{j,i} &= \{z = 0\} \times \Lambda_j \\ &\subset (N_{k,i} \setminus \text{Int}(N_{k-1,i})) \\ &\subset (\{t = 1\} \times (M \setminus \text{Int}(N_{k-1,1} \cup N_{k-1,2}))) \subset \partial W_{k-1} \end{aligned}$$

for  $i = 1, 2$ . Define the  $(k + 1)$ -dimensional disks

$$\tilde{\Lambda}_j = [-1, 1] \times \Lambda_j \subset (H_{\Sigma_k} \setminus \text{Int}(H_{\Sigma_{k-1}})) \subset (W_k \setminus \text{Int}(W_{k-1})).$$

The 1-form  $\beta$  vanishes on  $\Lambda = \{0\} \times \Lambda \subset [-\epsilon, \epsilon] \times \Sigma_k$  by the explicit description of the Weinstein handle given in Equation 2.3.1. Consequently, the Liouville 1-form  $\lambda = -\theta dz - 2zd\theta + \beta$  vanishes on  $\tilde{\Lambda}_j$ . The boundary of  $\tilde{\Lambda}_j$  is the piecewise smooth  $k$ -sphere

$$\partial \tilde{\Lambda}_j = \Lambda_{j,1} \cup ([-1, 1] \times \partial \Lambda_j) \cup \Lambda_{j,2}.$$

After smoothing the corners of  $W_{k-1}$ ,  $\tilde{\Lambda}_j \cap W_{k-1}$  will be a smooth isotropic sphere  $S_j^k$ , by the vanishing of the form  $\lambda$  along  $\tilde{\Lambda}$ . Noting that  $(W_k, \lambda_k)$  is obtained from  $(W_{k-1}, \lambda_{k-1})$  by attaching Weinstein handles along each of the  $S_j^k$ , the proof is complete.  $\square$

**7.3. Attaching handles to weak symplectic cobordisms and fillings.** In this section we describe when it is possible to attach a modified version of the symplectic handle  $(H_\Sigma, \omega_\Sigma)$  to the positive boundary of a weak symplectic cobordism  $(W, \omega)$ . The main result of this section is the following:

**Theorem 7.1.** *Let  $(M, \xi)$  be a  $(2n + 1)$ -dimensional contact manifold, which is the positive boundary of a weak symplectic cobordism  $(W, \omega)$  with negative boundary  $(M', \xi')$ . Let  $(\Sigma, \beta)$  be a  $2n$ -dimensional Liouville domain and let  $i_j : (\Sigma, \beta) \rightarrow (M, \xi)$ ,  $j = 1, 2$ , be Liouville embeddings with disjoint images. Assume that  $i_1^* \omega = i_2^* \omega$  in  $H^2(\Sigma; \mathbb{R})$ . Then there is a weak symplectic cobordism whose negative boundary is  $(M', \xi')$  and whose positive boundary is the manifold  $\#_{(\Sigma, \beta)}(M, \xi)$  obtained from  $(M, \xi)$  by a Liouville connect sum.*



To prove Theorem 7.1, we will need the following lemma, which summarizes some results appearing in [MNW11, §1]:

**Lemma 7.2.** *Suppose that  $(M, \xi)$  is the positive boundary of weak symplectic cobordism  $(W, \omega)$  and  $\alpha$  is a contact 1-form for  $(M, \xi)$ . Then  $W$  can be extended to a non-compact symplectic manifold  $(W', \omega')$  for which*

- (1)  $W' \setminus W$  is diffeomorphic to  $(0, \infty) \times M$ ,
- (2) the symplectic form  $\omega'$  coincides with  $\omega$  on  $W$ ,
- (3)  $\omega'|_{(t_0, \infty) \times M} = \omega|_{TM} + d(t\alpha)$  for a sufficiently large constant  $t_0 > 0$ , where  $t$  is a coordinate on  $(0, \infty)$ , and
- (4) each of the level sets  $(\{t\} \times M, \xi)$  is weakly filled for  $t > t_0$ .

*Proof of Theorem 7.1.* We continue to make use of the notation described in Section 7.1, modifying the construction described there as needed.

Suppose that  $\alpha$  is a contact form for  $(M, \xi)$  such that  $i_j^* \alpha = \beta$  for  $j = 1, 2$ . Consider the embeddings  $i_-|_{\{\theta=-1\}}, i_+|_{\{\theta=1\}} : \mathcal{N}(\Sigma) \rightarrow M$  defined in Step 3 of Section 7.1. By assumption, we have that  $i_-|_{\{\theta=-1\}}^* \omega = i_+|_{\{\theta=1\}}^* \omega$  as second cohomology classes in  $\mathcal{N}(\Sigma)$ . Therefore, we can find some  $\gamma \in \Omega^1(\mathcal{N}(\Sigma))$  such that

$$d\gamma = (i_+|_{\{\theta=1\}}^* - i_-|_{\{\theta=-1\}}^*)\omega|_{TM}.$$

Let  $f : [-1, 1] \rightarrow [0, 1]$  be a smooth function such that  $f(1) = 1$ ,  $f(0) = 0$ , and all derivatives of  $f$  are supported on  $[-\delta, \delta]$  for some  $\delta \in (0, 1)$ . Consider the family of 1-forms

$$\lambda_t := t\lambda + f(\theta)\gamma$$

on  $H_\Sigma = [-1, 1] \times \mathcal{N}(\Sigma)$ , for  $t > 0$ . Here  $\lambda$  is defined by Equation 7.1.1. It follows that

$$d\lambda_t = t \cdot d\lambda + \frac{\partial f}{\partial \theta} \cdot d\theta \wedge \gamma + f(\theta) \cdot d\gamma.$$

This implies that  $d\lambda_t$  is symplectic on  $H_\Sigma$  for  $t$  sufficiently large. Furthermore, we have that

$$d\lambda_t = \begin{cases} t \cdot d\lambda + i_-|_{\{\theta=-1\}}^* \omega|_{TM} & : \theta \in [-1, -1 + \delta] \\ t \cdot d\lambda + i_+|_{\{\theta=1\}}^* \omega|_{TM} & : \theta \in (1 - \delta, 1]. \end{cases}$$

Consider the non-compact symplectic manifold  $(W', \omega')$  constructed from the cobordism  $(W, \omega)$  as described in Lemma 7.2. Let  $t_1 > 0$  be an arbitrarily large constant and let  $(W'_{t_1}, \omega')$  be the compact symplectic manifold obtained by removing the open collar  $(t_1, \infty) \times M$  from  $W'$ . Attach the handle  $H_\Sigma$  to  $\partial W'_{t_1}$  as described in Step 3 of Section 7.1. By design, the 2-form  $\omega'$  agrees with  $d\lambda_{t_1}$  with respect to this gluing. Denote by  $(W'', \omega'')$  the symplectic manifold obtained by carrying out this gluing as well as the edge-rounding described in Step 4 of Section 7.1, where  $\omega''|_{W'_{t_1}} = \omega'$  and  $\omega''|_{H_\Sigma} = d\lambda_{t_1}$ .

For  $t_1 > t_0$  sufficiently large, we have that  $\omega''$  is symplectic on  $W''$ . Here  $t_0$  is the constant described in Lemma 7.2. Moreover, for an arbitrarily large  $t_1$ ,  $\frac{1}{t_1} \omega''|_{T(\partial W'')}$  is  $C^0$ -arbitrarily close to the exterior derivative of the contact 1-form  $\lambda|_{T\#(\Sigma, \beta)(M, \xi)}$  on  $\#(\Sigma, \beta)(M, \xi)$  described in Section 7.1. Therefore  $(W''_{t_1}, \omega''_{t_1})$  provides the desired weak cobordism for  $t_1$  sufficiently large.  $\square$

## 8. APPLICATIONS OF THEOREM 1.9

In this section we provide proofs of most of the applications of Theorem 1.9 stated in Section 1.5. The proof of each theorem will provide an example of a Liouville connect sum.

**8.1. Open books and mapping class monoids.** The purpose of this section is to prove Theorem 1.16.

The relationship between symplectomorphism groups of Liouville domains and contact manifolds established in Theorem 1.13 has attracted a great deal of interest, especially in dimension three. As an example, Baker-Etnyre-van Horn-Morris [BEV10, §1.2] and Baldwin [Ba10, Theorems 1.1 - 1.3] have shown that for a compact oriented surface  $\Sigma$  with  $\partial\Sigma \neq \emptyset$ , the contact manifolds supported by open books with page  $\Sigma$  which are fillable (in any of the senses of Definitions 2.4 and 2.6) constitute a monoid of  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$ . The Liouville connect sum and Theorem 1.9 provide a natural generalization of this result to open books whose pages are Liouville domains of any even dimension, as stated in Theorem 1.16.

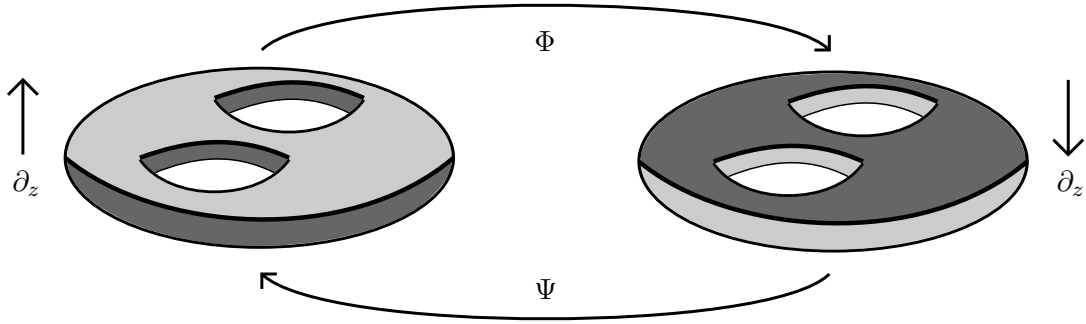


FIGURE 5. A Heegaard decomposition of a contact manifold  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  determined by an open book decomposition. The maps  $\Phi$  and  $\Psi$  provide instructions for performing a convex gluing as described in Section 3.6.

The first step in the proof of Theorem 1.16 is the following lemma, which allows us to translate open book descriptions of contact manifolds into Heegaard decomposition descriptions. See Figure 5.

**Remark 8.1.** *Our use of the expression “Heegaard decomposition” is, of course, informal when speaking of contact manifolds whose dimensions are greater than three.*

**Lemma 8.2.** *Let  $(N_j, \xi_{(\Sigma, \beta)})$  ( $j = 1, 2$ ) be two standard neighborhoods of a Liouville domain  $(\Sigma, \beta)$  and let  $\Phi, \Psi \in \text{Symp}((\Sigma, d\beta), \partial\Sigma)$ . Define the contact manifold  $(M, \xi)$  by the convex gluing instructions  $(\Phi, \Psi) : \partial N_1 \rightarrow \partial N_2$  so that  $(\Phi, \Psi)$  maps*

- (1)  $(\partial N_1)^+$  to  $(\partial N_2)^-$  via  $\Phi$  and
- (2)  $(\partial N_2)^+$  to  $(\partial N_1)^-$  via  $\Psi$ .

*See Figure 5. Then  $(M, \xi)$  is diffeomorphic to the contact manifold  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  determined by the pair  $((\Sigma, \beta), \Phi \circ \Psi)$ .*

*Proof.* This is a slight modification of the proof of Theorem 1.13(1). See [Et06, §3]. □

**Proposition 8.3.** *Let  $(\Sigma, \beta)$  be a Liouville domain and let  $\Phi, \Psi \in \text{Symp}((\Sigma, d\beta), \partial\Sigma)$ . Then  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  can be obtained from  $(M, \xi)_{((\Sigma, \beta), \Phi)} \sqcup (M, \xi)_{((\Sigma, \beta), \Psi)}$  by a Liouville connect sum.*

*Proof.* Let  $N_1, N_2, N'_1$  and  $N'_2$  be copies of a standard neighborhood of  $\Sigma$ , each endowed with the contact structure determined by the contact form  $dz + \beta$  as described in Section 3.1. By Lemma 8.2 we can construct  $(M, \xi)_{((\Sigma, \beta), \Phi)}$  by identifying  $\partial N_1$  and  $\partial N_2$  using the convex gluing instructions  $(\Phi, \text{id}_\Sigma)$ . Similarly, we can construct  $(M, \xi)_{((\Sigma, \beta), \Psi)}$  by identifying  $\partial N'_1$  and  $\partial N'_2$  using the convex gluing instructions  $(\Psi, \text{id}_\Sigma)$ .

Now perform a Liouville connect sum on  $(M, \xi) := (M, \xi)_{((\Sigma, \beta), \Phi)} \sqcup (M, \xi)_{((\Sigma, \beta), \Psi)}$  by removing  $N_2$  and  $N'_2$  and then identifying  $\partial N_1$  and  $\partial N'_1$ . Then the resulting contact manifold  $\#_{(\Sigma, \beta)}(M, \xi)$  can be described by identifying  $\partial N_1$  and  $\partial N'_1$  using the convex gluing instructions  $(\Phi, \Psi)$ . By Lemma 8.2, we have  $\#_{(\Sigma, \beta)}(M, \xi) = (M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$ . □

*Proof of Theorems 1.16 & 1.17.* (1) and (3) are immediate from Proposition 8.3, Theorem 1.9, and the fact that a composition of symplectic (exact, Stein) cobordisms is a symplectic (resp. exact, Stein) cobordism. For item (2) the cobordism  $(W, \lambda)$  from Theorem 1.9 can be obtained by a sequence of Weinstein handle attachments as both  $(M, \xi)_{((\Sigma, \beta), \Phi)}$  and  $(M, \xi)_{((\Sigma, \beta), \Psi)}$  are 3-dimensional. Regarding weak symplectic fillings, we note that the hypothesis of Theorem 1.17 guarantees that Theorem 7.1 can be applied, giving a weak symplectic filling of  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$ .  $\square$

**Remark 8.4.** *In the case  $\dim(\Sigma) = 2$ , the proof of Theorem 1.16 coincides with the proofs of [BEV10, Theorem 1.3] and [Ba10, Theorem 1.1]. This can be worked out by analyzing their proofs, the proof of Theorem 1.16, and the proof of the second statement of Theorem 1.9 appearing in Section 7.2. More intuition can be gained by reading Section 8.5.*

**8.2. More monoids from contact homology.** The purpose of this section is to interpret Theorem 1.16 algebraically using *contact homology*. This is an algebraic invariant of contact manifolds which is a part of the symplectic field theory (SFT) proposed by Eliashberg-Givental-Hofer in [EGH00]. Following a discussion regarding homomorphisms on contact homology induced by exact symplectic cobordisms, we prove Corollary 1.18. Throughout this section,  $(M, \xi)$  will be a closed contact manifold of dimension  $2n + 1$ . We begin by defining this invariant, following the exposition [Bo09].

**Remark 8.5.** *Below, we will define the contact homology with a canonical  $\mathbb{Z}/2\mathbb{Z}$  grading for closed contact manifolds. This definition has been extended to consider contact manifolds with convex boundary by Colin-Ghiggini-Honda-Hutchings in [CGHH10]. It should also be noted that the  $\mathbb{Z}/2\mathbb{Z}$  grading in our definition can be lifted to a  $\mathbb{Q}$ - or (when  $H_1(M; \mathbb{Z})$  is torsion free) a  $\mathbb{Z}$ -grading by using a suitably twisted coefficient system and taking into account additional geometric data. See [EGH00] for additional details.*

**Remark 8.6.** *At the time of the writing of this paper, the analytic foundations of symplectic field theory have not yet been rigorously established. These foundations are currently being developed as a part of the polyfold theory of Hofer-Wysocki-Zehnder [Hof04]. Consequently, we will ignore the issues of transversality and perturbations required to count holomorphic curves in asymptotically cylindrical symplectic manifolds. Therefore Theorem 1.18 should be regarded as conjecture, and its proof heuristic.*

**8.2.1. The algebra  $CC_*(M, \alpha)$ .** Fix a contact form  $\alpha$  for  $(M, \xi)$  which is *non-degenerate*. Non-degeneracy of  $\alpha$  means that for every parameterized, periodic orbit  $\gamma = \gamma(s) : [0, T_\gamma] \rightarrow M$  with period  $T_\gamma$  of the Reeb vector field  $R$  of  $\alpha$ , the Poincaré return map  $P_\gamma : \xi_{\gamma(0)} \rightarrow \xi_{\gamma(0)}$  associated to  $\gamma$  does not have 1 as an eigenvalue. Non-degeneracy of  $\alpha$  is a generic condition as is shown in [Bo09]. To every such orbit  $\gamma$  we may assign a *Conley-Zehnder index*  $CZ(\gamma) \in \mathbb{Z}/2\mathbb{Z}$ , defined by

$$(-1)^{CZ(\gamma)} = (-1)^n \operatorname{sgn} \circ \det(P_\gamma - \operatorname{id}_{\xi_{\gamma(0)}}).$$

From this definition, we say that the *grading*  $|\gamma|$  of the periodic orbit  $\gamma$  is

$$|\gamma| = CZ(\gamma) + n \in \mathbb{Z}/2\mathbb{Z}.$$

It ends up that the Conley-Zehnder index, and so grading, of  $\gamma$  is independent of parametrization. Therefore  $CZ$  is well defined on elements of the set of unparameterized periodic Reeb orbits for  $\alpha$ .

There are two ways that the Conley-Zehnder index can behave with respect to multiple coverings. For a periodic Reeb orbit  $\gamma$ , we have that either

- (1) all multiple coverings of  $\gamma$  have the same grading or
- (2) the gradings of even-fold coverings of  $\gamma$  disagree with the gradings of odd-fold coverings.

We say that a Reeb orbit  $\gamma$  is *bad* if it is an even-fold covering of a Reeb orbit  $\gamma'$  with  $|\gamma| \neq |\gamma'|$ . All other Reeb orbits are called *good*. We write  $\mathcal{P}_\alpha$  for the collection of good Reeb orbits of  $\alpha$ .

**Definition 8.7.** Define  $CC_*(\alpha)$  to be the unital, associative,  $\mathbb{Z}/2\mathbb{Z}$  graded, supercommutative algebra over  $\mathbb{Q}$ , freely generated by the elements of  $\mathcal{P}_\alpha$ .

Here, supercommutativity means that

$$\gamma_1 \cdot \gamma_2 = (-1)^{|\gamma_1| \cdot |\gamma_2|} \gamma_2 \cdot \gamma_1 \quad \forall \gamma_1, \gamma_2 \in \mathcal{P}_\alpha.$$

The unit is considered to coincide with the generator corresponding to the empty Reeb orbit,  $1 = \emptyset$ . The exclusion of the bad orbits from the definition of  $CC_*(M, \alpha)$  is required by the coherent orientation scheme for the moduli spaces of holomorphic curves used to define the differential in contact homology. See [BM04].

8.2.2. *The definition of  $\partial = \partial_{J, \nu, m}$ .* In order to define a differential on the algebra  $CC_*(\alpha)$ , we must first discuss markers on embedded Reeb orbits, almost complex structures, and certain moduli spaces of holomorphic curves.

To each  $\gamma \in \mathcal{P}_\alpha$  which is embedded, we choose a point  $m(\gamma)$  in its image. If  $\gamma$  is multiply covered, we define  $m(\gamma)$  to be the point on the image of  $\gamma$  in  $M$  assigned to the underlying embedded Reeb orbit which  $\gamma$  multiply covers. The  $m(\gamma)$  are called *markers*.

**Definition 8.8.** The symplectization of  $(M, \xi)$  is the non-compact symplectic manifold  $((0, \infty) \times M, d(t \cdot \alpha))$ .

For the remainder of this section, we will fix a non-degenerate contact form  $\alpha$  on  $(M, \xi)$ , markers  $m(\gamma)$  on each of the Reeb orbits of  $\alpha$ , and a choice of almost complex structure  $J$  adapted to the symplectization of  $(M, \xi)$ , as described in Example 2.5.

**Definition 8.9.** A decorated, rational, Riemann surface with one positive puncture is a  $(k+1)$ -punctured sphere  $\mathbb{CP}^1 \setminus \{x^+, x_1^-, \dots, x_k^-\}$ , endowed with its unique complex structure  $j$  together with a choice of an asymptotic marker for each puncture, defined as follows: Each puncture  $x$  admits a neighborhood conformally equivalent to  $[0, \infty) \times S^1$  equipped with its unique complex structure. An asymptotic marker for  $x$ , denoted  $m(x)$  is a choice of a point  $m(x)$  on the boundary  $S^1 \times \{\infty\}$  of the compactification  $S^1 \times [0, \infty]$  of this neighborhood.

The moduli space of such Riemann surfaces (and its compactification) is discussed in [BEHWZ03, §4]. For each puncture  $x_j$  in a decorated, rational, Riemann surface we take coordinates  $(\theta, s)$  on a neighborhood  $N_{x_j} = [0, \infty) \times S^1$  of a puncture  $x_j$  as discussed above.

**Definition 8.10.** A decorated, rational, finite energy,  $J$ -holomorphic curve with one positive puncture in  $(M, \xi)$  is a map  $(a, u)$  from a decorated, rational, Riemann surface  $\mathbb{CP}^1 \setminus \{x^+, x_1^-, \dots, x_k^-\}$  into the symplectization of  $(M, \xi)$  such that

- (1)  $J \circ Tu = Tu \circ j$ ,
- (2) at the positive puncture  $x^+$ , there is some  $T \in (0, \infty)$  such that in a neighborhood  $N_{x^+}$

$$a \rightarrow \infty \quad \text{as} \quad s \rightarrow \infty, \quad \text{and} \quad \lim_{s \rightarrow \infty} (u(s, T \cdot *))$$

converges in  $\mathcal{C}^\infty(S^1, M)$  to a Reeb orbit  $\gamma^+$  in  $(M, \xi)$ , with  $m(x^+)$  being sent to  $m(\gamma^+)$ ,

- (3) for each negative puncture  $x_j^-$ , there is some  $T \in (-\infty, 0)$  such that in a neighborhood  $N_{x_j^-}$

$$a \rightarrow -\infty \quad \text{as} \quad s \rightarrow \infty, \quad \text{and} \quad \lim_{s \rightarrow \infty} (u(s, T \cdot *))$$

converges in  $\mathcal{C}^\infty(S^1, M)$  to a Reeb orbit  $\gamma_j^-$  in  $(M, \xi)$ , with  $m(x_j^-)$  being sent to  $m(\gamma_j^-)$ , and

- (4)  $(a, u)$  has finite Hofer energy, as described in [BEHWZ03, §5.3].

Now, given some  $\gamma^+ \in \mathcal{P}_\alpha$  and a collection  $\{\gamma_1^-, \dots, \gamma_k^-\}$  of Reeb orbits in  $\mathcal{P}_\alpha$  we can define a moduli space of  $J$ -holomorphic curves denoted

$$\mathcal{M}_J(\gamma^+ : \gamma_1^-, \dots, \gamma_k^-)$$

to be space of  $J$ -holomorphic maps  $(a, u)$  from rational, decorated, Riemann surfaces with one positive punctures and  $k$  negative punctures in  $(M, \xi)$ , sending a positive puncture to the  $\gamma^+$  and the negative punctures to the  $\gamma_j^-$  as in the above definition, modulo biholomorphic reparameterization of the domain. This space comes equipped with a  $\mathbb{R}$ -action, given by translating the image of a holomorphic map  $(a, u)$  in the symplectization of  $(M, \xi)$  using the vector field  $t\partial_t$ . From this action we can also consider the spaces

$$\mathcal{M}_J(\gamma^+ : \gamma_1^-, \dots, \gamma_k^-)/\mathbb{R}.$$

To each  $(a, u) \in \mathcal{M}_J(\gamma^+ : \gamma_1^-, \dots, \gamma_k^-)$ , we can assign a *Fredholm index* which under suitable hypotheses measures the dimension of  $\mathcal{M}_J(\gamma^+ : \gamma_1^-, \dots, \gamma_k^-)$  near  $(a, u)$ . After modding out by the  $\mathbb{R}$  action, the collection of points in  $\mathcal{M}_J(\gamma^+ : \gamma_1^-, \dots, \gamma_k^-)/\mathbb{R}$  corresponding to Fredholm index 1 curves is finite and can be used to produce a rational number

$$\begin{aligned} \eta(\gamma^+ : \gamma_1^- \cdots \gamma_k^-) &\in \mathbb{Q} \quad \text{satisfying} \\ \eta(\gamma^+ : \gamma_1^- \cdots \gamma_{j+1}^- \gamma_j^- \cdots \gamma_k^-) &= (-1)^{|\gamma_j^-| \cdot |\gamma_{j+1}^-|} \eta(\gamma^+ : \gamma_1^- \cdots \gamma_k^-). \end{aligned}$$

The definition of this number takes into account covering multiplicities of the Reeb orbits involved, the number of times an orbit  $\gamma_j^-$  occurs in the list  $\{\gamma_1^-, \dots, \gamma_k^-\}$ , and requires the choice  $\nu$  of an orientation on the moduli spaces in order to count curves algebraically. See [BM04] and [EGH00, §2].

Using the number defined in Equation 8.2.2, we can define the contact homology differential  $\partial = \partial_{J, \nu, m}$ . For a Reeb orbit  $\gamma \in \mathcal{P}_\alpha$  we define

$$\partial\gamma = \sum \eta(\gamma : \gamma_1 \cdots \gamma_k) \cdot \gamma_1 \cdots \gamma_k$$

where the sum is taken over all ordered collections  $\{\gamma_1, \dots, \gamma_k\}$  of Reeb orbits in  $\mathcal{P}_\alpha$  satisfying  $|\gamma_1 \cdots \gamma_k| = |\gamma| - 1$ . The differential is then extended to homogeneous elements  $x, y \in CC_*(\alpha)$  by the Leibnitz rule

$$\partial(xy) = (\partial x) \cdot y + (-1)^{|x|} x \cdot (\partial y),$$

and then extended linearly over  $\mathbb{Q}$ .

**Theorem 8.11** ([EGH00]). *The differential  $\partial$  satisfies  $\partial^2 = 0$  and its homology is an invariant of the contact manifold  $(M, \xi)$ . Therefore, we define the contact homology algebra  $HC_*(M, \xi, \mathbb{Q})$  of  $(M, \xi)$  as the homology of  $(CC_*(\alpha), \partial_{J, \nu, m})$ .*

We can analogously construct modified versions of contact homology by counting rational holomorphic curves with multiple positive punctures (defining the *rational SFT algebra*) or by counting holomorphic curves of all genera with an arbitrary number of positive and negative punctures (defining the *full SFT algebra*). See [EGH00]. However, as shown in [BN10], the vanishing of any of these invariants is equivalent to the vanishing of the contact homology.

As noted Eliashberg and Yau in [Y06], all 3-dimensional contact manifolds which are overtwisted (in the usual sense) are algebraically overtwisted. In fact, at the time of the writing of this paper, there is no known example of a closed tight contact 3-manifold which is algebraically overtwisted. Niederkrüger and Wendl observe in [NW11, Corollary 6] that symplectically fillable contact manifolds always have non-vanishing contact homology (as defined above) and that weakly fillable contact 3-manifolds always have non-vanishing contact homology when computed using some twisted coefficient system.

8.2.3. *Homomorphisms from exact symplectic cobordisms.* Now suppose that  $(M, \xi)$  and  $(M', \xi')$  are contact manifolds of the same dimension. In this section we describe how an exact symplectic cobordism  $(W, \lambda)$  from  $(M', \xi')$  to  $(M, \xi)$  determines a homomorphism

$$\Phi^{(W, \lambda)} : HC_*(M, \xi, \mathbb{Q}) \rightarrow HC_*(M', \xi', \mathbb{Q}).$$

Suppose that  $\lambda|_M = \alpha$  and that  $\lambda|_{M'} = \alpha'$  for some contact forms  $\alpha$  on  $(M, \xi)$  and  $\alpha'$  on  $(M', \xi')$ . Then there exist a collar neighborhood of  $\partial W$  of the form

$$([1/2, 1] \times M) \cup ([1/2, 1] \times M') \quad \text{with} \quad \partial W = (\{1\} \times M) \sqcup -(\{1/2\} \times M')$$

where  $\lambda|_{[1/2, 1] \times M} = t \cdot \alpha$  and  $\lambda|_{[1/2, 1] \times M'} = t \cdot \alpha'$ .

We can then define the *completion*  $\overline{(W, \lambda)} = (\overline{W}, \overline{\lambda})$  of  $(W, \lambda)$  to be the non-compact, exact symplectic manifold

$$\overline{W} = W \cup ([1, \infty) \times M) \cup ((0, 1/2] \times M') \quad \text{where}$$

$$\overline{\lambda}|_{[1, \infty) \times M} = t \cdot \alpha \quad \text{and} \quad \overline{\lambda}|_{(0, 1/2] \times M'} = t \cdot \alpha'.$$

As in Definition 8.8, we say that an almost complex structure  $J$  on  $\overline{(W, \lambda)}$  is *adapted to the completion* if  $d\lambda(*, J*)$  is a Riemannian metric and it satisfies the conditions of Example 2.5 on each of  $[1, \infty) \times M$  and  $(0, 1/2] \times M'$ . Similarly, assuming non-degeneracy of both  $\alpha$  and  $\alpha'$ , given  $\gamma^+ \in \mathcal{P}_\alpha$  and a collection  $\{\gamma_1^-, \dots, \gamma_k^-\} \subset \mathcal{P}_{\alpha'}$ , together with choices of markers for all relevant orbits, we can modify Definition 8.10 in the obvious way as to define *decorated, rational, finite energy,  $J$ -holomorphic curves with one positive puncture* in  $\overline{(W, \lambda)}$ . Therefore, for each  $\gamma^+ \in \mathcal{P}_\alpha$  and of collection of orbits  $\{\gamma_1^-, \dots, \gamma_k^-\} \subset \mathcal{P}_{\alpha'}$ , we may define a moduli space of  $J$ -holomorphic curves, denoted

$$\mathcal{M}_J^W(\gamma^+ : \gamma_1^-, \dots, \gamma_k^-)$$

to be the space of all decorated, rational, finite energy  $J$ -holomorphic curves in  $\overline{(W, \lambda)}$  with one positive punctures asymptotic to  $\gamma^+$ , and negative punctures asymptotic to the  $\gamma_j^-$ , modulo biholomorphic reparameterization.

In this situation, there is not necessarily an  $\mathbb{R}$ -action on these moduli spaces as in the case of  $J$ -holomorphic curves in a symplectization. Therefore, instead of counting Fredholm index 1 curves in  $\overline{(W, \lambda)}$  we count index 0 curves.

Because our symplectic cobordism  $(W, \lambda)$  is exact, the energies [BEHWZ03, §6] of the holomorphic curves in a moduli space  $\mathcal{M}_J^W(\gamma^+ : \gamma_1^-, \dots, \gamma_k^-)$  are uniformly bounded by a constant multiple of the period of the Reeb orbit  $\gamma^+ \in \mathcal{P}_\alpha$ . Moreover, the holomorphic curves in  $\mathcal{M}_J^W(\gamma_1^+ : \gamma_1^-, \dots, \gamma_k^-)$  are *action decreasing*. This means that if this moduli space is non-empty, we must have

$$\int_{\gamma^+} \alpha \geq \sum_1^k \int_{\gamma_j^-} \alpha'.$$

Therefore, we can apply the SFT compactness theorem [BEHWZ03, Theorem 10.2] to conclude that there are only a finite number of holomorphic curves (up to biholomorphic reparameterization) in  $\overline{(W, \lambda)}$  of Fredholm index 0 with a single positive puncture asymptotic to a fixed Reeb orbit  $\gamma^+ \in \mathcal{P}_\alpha$ . As was the case in Equation 8.2.2, this allows us to define a rational number

$$\eta^{(W, \lambda)}(\gamma^+, \gamma_1^- \dots \gamma_k^-) \in \mathbb{Q}$$

for each  $\gamma^+ \in \mathcal{P}_\alpha$  and ordered collection  $\{\gamma_1^-, \dots, \gamma_k^-\} \subset \mathcal{P}_{\alpha'}$  by counting the Fredholm index 0 curves algebraically. As in the case of Equation 8.2.2 the definition of this number depends on a choice  $\nu$  of orientations for all of the relevant moduli spaces and must take into account the covering multiplicities and number of occurrences of each  $\gamma_j^-$  in the list.

Now for a Reeb orbit  $\gamma \in \mathcal{P}_\alpha$  we define

$$\Phi^{(W,\lambda)}\gamma = \sum \eta^{(W,\lambda)}(\gamma, \gamma_1 \dots \gamma_k) \cdot \gamma_1 \dots \gamma_k$$

where the sum runs over all unordered collections  $\{\gamma_1, \dots, \gamma_k\}$  of Reeb orbits in  $\mathcal{P}_{\alpha'}$  satisfying  $|\gamma| = |\gamma_1 \dots \gamma_k|$ .

By the fact that holomorphic curves in  $(\overline{W}, \lambda)$  are action decreasing and that  $\alpha'$  is assumed to be non-degenerate, we have that for a fixed  $\gamma \in \mathcal{P}_\alpha$  the moduli spaces  $\mathcal{M}_J^{(W,\lambda)}(\gamma : \gamma_1^-, \dots, \gamma_k^-)$  are empty - and so the numbers  $\eta^{(W,\lambda)}(\gamma, \gamma_1 \dots \gamma_k)$  are zero - for all but finitely many collections  $\{\gamma_1, \dots, \gamma_k\}$ . Note also that by Stokes' Theorem, every non-constant holomorphic curve in  $(\overline{W}, \lambda)$  must have a positive puncture. Therefore we can define  $\Phi^W(1) = \Phi^W(\emptyset) = 1 \in CC_*(M', \alpha')$  so that the map  $\Phi^{(W,\lambda)}$  can naturally be extended to a homomorphism of unital algebras

$$\Phi^{(W,\lambda)} : CC_*(M, \alpha) \rightarrow CC_*(M, \alpha').$$

Counting broken holomorphic curves (i.e. height-2 holomorphic buildings in  $(\overline{W}, \lambda)$  split along each of one of either  $M$  or  $M'$ ) in the boundaries of the compactified moduli spaces of Fredholm index 1 holomorphic curves in  $\overline{W}$  yields the equation

$$\partial_{CC_*(M', \alpha')} \circ \Phi^{(W,\lambda)} = \Phi^{(W,\lambda)} \circ \partial_{CC_*(M, \alpha)}.$$

Morally speaking, the difference of the left and right hand sides of the above equations, evaluated at any  $x \in CC_*(M, \alpha)$ , is a signed count of the number of points in the boundary of a compact 1-dimensional manifold, and so is zero. We conclude that  $\Phi^{(W,\lambda)}$  descends to a homomorphism on the level of homology

$$(8.2.1) \quad \Phi^{(W,\lambda)} : HC_*(M, \xi, \mathbb{Q}) \rightarrow HC_*(M', \xi', \mathbb{Q}).$$

**Remark 8.12.** Equation 8.2.1 is essentially [EGH00, Theorem 2.3.7], where the hypothesis that  $W$  is a rational homology cobordism is removed at the expense of being able to only consider untwisted coefficient systems.

**Theorem 8.13.** Suppose that  $(W, \lambda)$  is an exact symplectic cobordism from  $(M', \xi')$  to  $(M, \xi)$  and that  $(M', \xi')$  has non-vanishing contact homology. Then  $(M, \xi)$  also has non-vanishing contact homology.

*Proof.* Suppose that  $HC_*(M, \xi, \mathbb{Q}) = 0$ . This is equivalent to the statement that there is some  $x \in CC_*(M, \alpha)$  satisfying  $\partial(x) = 1$ . It follows that

$$\partial_{CC_*(M', \alpha')}(\Phi^{(W,\lambda)}(x)) = \Phi^{(W,\lambda)}(\partial_{CC_*(M, \alpha)}(x)) = \Phi^{(W,\lambda)}(1) = 1 \in CC_*(M', \alpha').$$

We conclude that the unit in  $CC_*(M', \alpha')$  is exact. This is equivalent to saying that  $HC_*(M', \xi', \mathbb{Q}) = 0$ , contradicting our hypothesis.  $\square$

**8.2.4. Proof of Corollary 1.18.** In this section we complete the proof of Corollary 1.18.

Suppose that  $A = A_0 \oplus A_1$  and  $B = B_0 \oplus B_1$  are  $\mathbb{Z}/2\mathbb{Z}$ -graded, supercommutative algebras over  $\mathbb{Q}$ . Define their *graded tensor product*  $A \hat{\otimes} B$  to be the  $\mathbb{Z}/2\mathbb{Z}$ -graded algebra for which

$$(A \hat{\otimes} B)_0 = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1), \quad (A \hat{\otimes} B)_1 = (A_1 \otimes B_0) \oplus (A_0 \otimes B_1),$$

and multiplication is defined by linearly extending the equation

$$(a \otimes b) \cdot (a' \otimes b') := (-1)^{|b| \cdot |a'|} (a \cdot a') \otimes (b \cdot b')$$

defined for all homogeneous elements  $a, a' \in A$  and  $b, b' \in B$ . Here all tensor products are taken over  $\mathbb{Q}$ .

**Proposition 8.14.** *Suppose that  $(M, \xi)$  and  $(M', \xi')$  are  $(2n + 1)$ -dimensional contact manifolds. Then*

$$HC(M \sqcup M', \xi \sqcup \xi', \mathbb{Q}) \cong HC(M, \xi, \mathbb{Q}) \hat{\otimes} HC(M', \xi', \mathbb{Q})$$

*as graded algebras.*

*Proof.* Suppose that  $\alpha$  and  $\alpha'$  are non-degenerate contact forms for  $(M, \xi)$  and  $(M', \xi')$  respectively. Clearly,  $\alpha \sqcup \alpha'$  is then a non-degenerate contact form for the disjoint union. Write  $(CC_*(\alpha), \partial_\alpha)$ ,  $(CC_*(\alpha'), \partial_{\alpha'})$ , and  $(CC_*(\alpha \sqcup \alpha'), \partial_{\alpha \sqcup \alpha'})$  for the differential graded algebras which compute  $HC_*(M, \xi, \mathbb{Q})$ ,  $HC_*(M', \xi', \mathbb{Q})$ , and  $HC_*(M \sqcup M', \xi \sqcup \xi', \mathbb{Q})$ , respectively.

It amounts to a change in notation that

$$(8.2.2) \quad CC_*(\alpha \sqcup \alpha') = CC_*(\alpha) \hat{\otimes} CC_*(\alpha')$$

as graded, supercommutative algebras. Indeed, given a monomial of non-trivial Reeb orbits  $\gamma_1 \cdots \gamma_k \in CC_*(\alpha \sqcup \alpha')$ , there is some  $I \subset \{1, \dots, k\}$  such that  $j \in I$  implies that  $\gamma_j$  is a closed Reeb orbit in  $(M, \alpha)$  and  $j \in \{1, \dots, k\} \setminus I$  implies that  $\gamma_j$  is a closed Reeb orbit in  $(M', \alpha')$ . If  $I$  is a proper, non-empty subset of  $\{1, \dots, k\}$ , we can write

$$\gamma_1 \cdots \gamma_k = \pm \left( \prod_{j \in I} \gamma_j \right) \cdot \left( \prod_{j \in \{1, \dots, k\} \setminus I} \gamma_j \right) \sim \pm \left( \prod_{j \in I} \gamma_j \right) \otimes \left( \prod_{j \in \{1, \dots, k\} \setminus I} \gamma_j \right).$$

If  $I = \emptyset$ , then we write  $\gamma_1 \cdots \gamma_k \sim 1_\alpha \otimes (\gamma_1 \cdots \gamma_k)$  where  $1_\alpha$  denotes the unit in  $CC_*(\alpha)$ . If  $I = \{1, \dots, k\}$ , we write  $\gamma_1 \cdots \gamma_k \sim (\gamma_1 \cdots \gamma_k) \otimes 1_{\alpha'}$  where  $1_{\alpha'}$  denotes the unit in  $CC_*(\alpha')$ .

The  $\mathbb{Q}$ -linear map determined by the symbol “ $\sim$ ” in the above equations defines a vector space isomorphism as in Equation 8.2.2. It then follows from supercommutativity of all algebras involved that “ $\sim$ ” is an algebra isomorphism. The unit in  $CC_*(\alpha \sqcup \alpha')$  then corresponds to  $1_\alpha \otimes 1_{\alpha'}$ .

Using the Leibnitz rule for  $\partial_{\alpha \sqcup \alpha'}$  together with the facts that

- (1)  $|1_\alpha| = |1_{\alpha'}| = 0$ ,
- (2)  $\partial_\alpha 1_\alpha = 0$ ,  $\partial_{\alpha'} 1_{\alpha'} = 0$ , and
- (3)  $\partial_{\alpha \sqcup \alpha'}$  sends Reeb orbits in  $M$  ( $M'$ ) to products of Reeb orbits in  $M$  ( $M'$ ), respectively) by the connectedness of the holomorphic curves involved

that we can write  $\partial_{\alpha \sqcup \alpha'}$  in graded tensor product notation as

$$\partial_{\alpha \sqcup \alpha'}(x \otimes y) = (\partial_\alpha x) \otimes y + (-1)^{|x|} x \otimes (\partial_{\alpha'} y)$$

for all pairs of homogeneous elements  $x \in CC_*(\alpha)$  and  $y \in CC_*(\alpha')$ . Therefore the proposition follows from the Künneth formula.  $\square$

Now we finish the proof of Corollary 1.18. Suppose as in the statement of the corollary that  $(\Sigma, \beta)$  is a Liouville domain and  $\Phi, \Psi \in \text{Symp}((\Sigma, d\beta), \partial\Sigma)$  are such that the contact manifolds  $(M, \xi)_{((\Sigma, \beta), \Phi)}$  and  $(M, \xi)_{((\Sigma, \beta), \Psi)}$  both have non-vanishing contact homology with coefficient ring  $\mathbb{Q}$ . By Proposition 8.14 the disjoint union  $(M, \xi)_{((\Sigma, \beta), \Phi)} \sqcup (M, \xi)_{((\Sigma, \beta), \Psi)}$  also has non-vanishing contact homology. Then applying Theorem 8.13 to the cobordism provided by Proposition 8.3 and Theorem 1.9, the contact homology of  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  is also non-zero.

**8.3. Contact manifolds which fiber over the circle.** In this section we introduce a family of contact manifolds which fiber over the circle  $S^1$  and discuss their symplectic fillability as described in Theorem 1.19.



**8.3.1. Definition of  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$ .** Let  $(\Sigma, \beta)$  be a  $2n$ -dimensional Liouville domain and let  $\Phi, \Psi \in \text{Symp}((\Sigma, d\beta), \partial\Sigma)$ . Consider the model neighborhood  $(\mathcal{N}(\Sigma), \text{Ker}(dz + \beta))$  of  $\Sigma$  and the associated  $[-1, 1]$ -invariant contact structure  $\text{Ker}(\alpha) = \text{Ker}(\hat{\alpha})$  on  $[-1, 1] \times \partial\mathcal{N}(\Sigma)$  described in Section 3.3. Here we have set the constant  $\delta$  from Section 3.3 equal to 1 for simplicity. Clearly  $([-1, 1] \times \partial\mathcal{N}(\Sigma), \text{Ker}(\alpha))$  has a convex boundary whose components can be identified to produce a closed contact manifold.

**Definition 8.15.** The  $(2n + 1)$ -dimensional contact manifold  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$  is obtained gluing together the boundary components of  $[-1, 1] \times \partial\mathcal{N}(\Sigma)$  together using the convex gluing instructions  $(\Phi, \Psi)$  as described in Section 3.6. See Figure 2.

It follows from Proposition 3.11 that  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$  depends only on the isotopy classes of  $\Phi$  and  $\Psi$  in  $\text{Symp}((\Sigma, d\beta), \partial\Sigma)$ .

**Remark 8.16.** The contact manifolds  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$  can also be described using supporting open books and the contact fiber sum [Ge97, §3]. Perform a contact fiber sum of the contact manifolds  $(M, \xi)_{((\Sigma, \beta), \Phi)}$  and  $(M, \xi)_{((\Sigma, \beta), \Psi)}$  along the bindings of their associated open books, using the pages of the open books to frame the relevant normal bundles. The resulting contact manifold will be  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$ . In dimension three, this is an example of the “blown-up, summed open book” construction described in [Wen10].

**8.3.2. Fillability of  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$ .** Having defined the  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$ , we are ready to prove Theorem 1.19. As in the proof of Theorem 1.16, we consider Heegaard splitting-type decompositions of contact manifolds determined by open books. This time, instead of Liouville connect summing pages of *distinct* open books, we apply the Liouville connect sum to the interiors of two pages of the *same* open book.

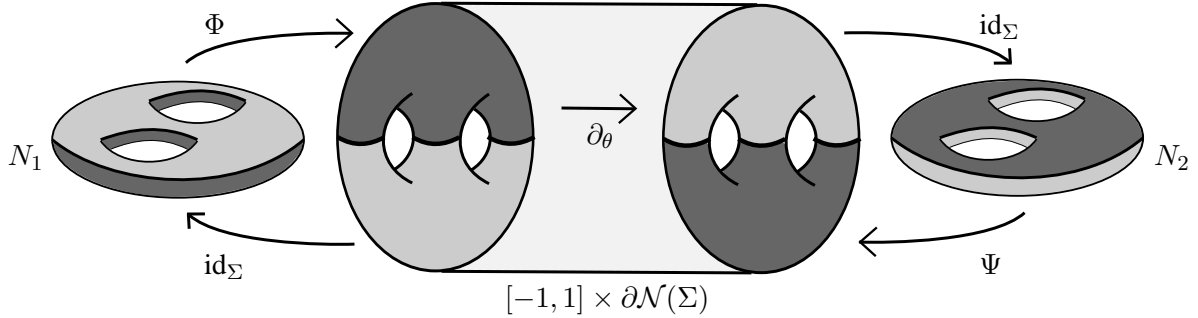


FIGURE 6. A decomposition of the contact manifold  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$  into three pieces with convex gluing instructions.

**Lemma 8.17.** Let  $(\Sigma, \beta)$  be a Liouville domain and let  $\Phi, \Psi \in \text{Symp}((\Sigma, d\beta), \partial\Sigma)$ . The contact manifold  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$  can be obtained from a contact manifold supported by an open book determined by the pair  $((\Sigma, \beta), \Phi \circ \Psi)$  by a Liouville connect sum.

*Proof.* We decompose the contact manifold  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  into three pieces  $[-1, 1] \times \partial\mathcal{N}(\Sigma)$ ,  $N_1$  and  $N_2$ . This decomposition will be a slight modification of the Heegaard splitting decomposition used in Lemma 8.2.

Take  $N_1$  and  $N_2$  to be standard neighborhoods of the Liouville domain  $(\Sigma, \beta)$ . Attach  $\{-1\} \times \partial\mathcal{N}(\Sigma)$  to  $\partial N_1$  using the convex gluing instructions  $(\text{id}_\Sigma, \Phi)$ . Similarly, attach  $\{1\} \times \partial\mathcal{N}(\Sigma)$  to  $\partial N_2$  using the convex gluing instructions  $(\text{id}_\Sigma, \Psi)$ . The resulting contact manifold is  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  as can be seen from Lemma 8.2. See Figure 6.

Now perform a Liouville connect sum on  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  along the standard neighborhoods of Liouville hypersurfaces  $N_1$  and  $N_2$ . This may be done by removing  $N_1$  and  $N_2$  from  $(M, \xi)_{((\Sigma, \beta), \Phi \circ \Psi)}$  and gluing

together the new convex boundary components. By the identifications described in the previous paragraph, the resulting contact manifold is exactly  $(M, \xi)_{((\Sigma, \beta), \Phi, \Psi)}$  as described in Definition 8.15.  $\square$

Together with Theorem 1.9, the above lemma immediately proves Theorem 1.19(1-3). As for the statement regarding weak symplectic fillings, we must show that the cohomological condition described in the statement of Theorem 1.19 coincides with the one described in the statement of Theorem 7.1. We observe that the Liouville embeddings required to perform the necessary symplectic handle attachment must agree with the submanifolds  $N_1$  and  $N_2$  in the above proof. Note that if we isotop  $N_2$  through the region  $[-1, 1] \times \mathcal{N}(\Sigma)$  and into  $N_1$  counterclockwise through the diagram shown in Figure 6, we see that  $\omega|_{N_2} = (\Phi^{-1})^* \omega|_{N_1}$  giving the cohomological obstruction described in Theorem 1.19 whose vanishing is required by Theorem 7.1.

**8.4. Fillability of branched covers.** In this section we apply Theorem 1.9 to study branched covers of contact manifolds. The following proposition, due to Geiges [Ge97] generalizing a construction of Gonzalo [Gon87, §2], describes a method of constructing contact manifolds by branched covering.

**Proposition 8.18.** *Let  $(M, \xi)$  be a  $(2n + 1)$ -dimensional contact manifold and let  $(C, \zeta) \subset (M, \xi)$  be a closed, codimension two contact submanifold with trivial normal bundle. Let  $\pi : \widetilde{M} \rightarrow M$  be a branched cover of  $M$  with branch locus  $C$ . Then  $\widetilde{M}$  naturally carries a contact structure  $\xi_\pi$  for which the associated unbranched covering  $\pi : (\widetilde{M} \setminus \pi^{-1}(N(C)), \xi_\pi) \rightarrow (M \setminus N(C), \xi)$  satisfies  $T\pi(\xi_\pi) = \xi$  where  $N(C)$  is an arbitrarily small tubular neighborhood of  $C$ .*

*Proof.* For simplicity, assume that  $C$  is connected and that the branching index about  $C$  is  $q > 0$ . After fixing a trivialization of the normal bundle of  $C$ , we can identify a tubular neighborhood  $N(C)$  of  $C$  with

$$N(C) = C \times \mathbb{D}^2 \quad \text{with} \quad \xi|_{N \times \mathbb{D}^2} = \text{Ker}(\alpha_C + r^2 d\theta)$$

where  $\alpha_C$  is a contact form for  $(C, \zeta)$  and  $(r, \theta)$  are polar coordinates on  $\mathbb{D}^2$ . This is a consequence of the Darboux theorem. The branched covering  $\pi$  restricts to an unbranched cover on the complement of  $T \times \mathbb{D}^2$  and so we may pull back  $\xi$  to  $\widetilde{M} \setminus \pi^{-1}(N(C))$  without difficulty. However, on  $\pi^{-1}(C \times \mathbb{D}^2)$

$$\pi(x, z) = (\pi(x), z^q) \quad \text{so that} \quad \pi^* \xi|_{C \times \mathbb{D}^2} = \text{Ker}(\pi^* \alpha_C + r^{2q} d\theta).$$

As  $\pi|_{\pi^{-1}(C)}$  is an unbranched cover, the 1-form  $\pi^* \alpha_C$  is a contact form. However the above equation indicates that  $\pi^*(\xi)$  is not contact on  $\pi^{-1}(N(C))$ , so that the contact structure and branched covering are incompatible near the branch locus.

To correct this, we can find a function  $f = f(r) : [0, 1] \rightarrow \mathbb{R}$  such that

- (1)  $f(r) = r^2$  near  $r = 0$ ,
- (2)  $f(r) = r^{2q}$  near  $r = 1$ , and
- (3)  $f'(r) > 0$  on  $(0, 1]$ .

Then we can define  $\xi_\pi$  on  $\pi^{-1}(C) \times \mathbb{D}^2 \subset \widetilde{M}$  by

$$\text{Ker}(\pi^* \alpha_C + f(r) d\theta).$$

To complete the proof, one must show that  $\xi_\pi$  is independent of the framing of the normal bundle  $N(C)$ , the size of the tubular neighborhood  $N(C)$ , and the function  $f$ . These facts can easily be established by Moser's argument [MS99, §3.2].  $\square$

**Definition 8.19.** *We write the cyclic cover of  $(M, \xi)$ , of branch index  $q$ , branched over the codimension two contact submanifold  $(C, \zeta)$  as  $(M, \xi)_{C, q}$ .*

Here we give some examples of known results regarding branched coverings of contact 3-manifolds.

**Theorem 8.20.** *Let  $(M, \xi)$  be a 3-dimensional contact manifold containing the transverse link  $C$ .*

- (1) [Ba10] *If  $C$  is a knot which realizes its Bennequin bound (Equation 6.2.1), then there is a Stein cobordism from  $(M, \xi)$  to  $(M, \xi)_{C,q}$ .*
- (2) [Gi02, MM91]  *$(M, \xi)$  can be described as a (not necessarily cyclic) branched cover over a transverse link in  $(S^3, \xi_{std})$ .*
- (3) [HPK09] *Suppose  $(M, \xi) = (S^3, \xi_{std})$ . For any  $q$  the contact distribution on  $(M, \xi)_{C,q}$  satisfies  $c_1(\xi_{C,q}) = 0$ , and its homotopy class depends only on the self-linking number and topological type of  $C$ . If  $C$  is destabilizeable, then a cyclic branched cover over  $C$  is overtwisted. If  $C$  can be represented as a quasipositive braid, then a cyclic branched cover over  $C$  is Stein fillable.*

Using Liouville sums we can find results (Theorem 1.20) similar to those concerning Stein fillings in items (1) and (3) of Theorem 8.20 which hold for contact manifolds of arbitrary dimension.

*Proof of Theorem 1.20.* We will show that we can obtain  $(M, \xi)_{C,q}$  from  $\sqcup^q(M, \xi)$  by performing a series of Liouville connect sums. Then the results regarding exact and Stein cobordisms will follow from Theorem 1.9. The last statement regarding weak symplectic fillings follows from the fact that the cohomological obstruction described in Theorem 7.1 always vanishes in this context, allowing for the desired handle attachments to the boundary of  $q$  copies of a weak filling of  $(M, \xi)$ .

The way to construct the topological manifold  $\widetilde{M}_{C,q}$  via cut and paste - i.e. without directly appealing to covering space theory - is as follows:

- (1) Identify a neighborhood of  $\Sigma$  with  $[0, \epsilon] \times \Sigma$  so that  $C = \{0\} \times \partial\Sigma$ .
- (2) Write  $N_j$  for the set  $[(j-1)/q, j/q] \times \Sigma \subset [0, \epsilon] \times \Sigma$  for  $j = 1, \dots, (q-1)$ .
- (3) Consider  $(q-1)$  additional copies of  $M$ , labeled  $M_j$ . Define  $N'_j = [0, 1/q] \times \Sigma \subset M_j$ .
- (4) Define  $M^0 = M$ . Inductively define  $M^j = (M^{j-1} \setminus N_j) \cup_{\Phi_j} (M_j \setminus N'_j)$  where  $\Phi_j : \partial N'_j \rightarrow \partial N_j$  is given by

$$\Phi_j(z, x) = ((j-1)/q - z, x) \quad \text{for } z \in [0, 1/q] \quad \text{and } x \in \Sigma.$$

Here we are considering  $N_j$  as a subset of  $M^{j-1}$  in the above.

See, for example, the construction of cyclic branched covers, branched over null-homologous links in 3-manifolds described in [R90, §5.C]. This easily carries over to branched covers, branched over null-homologous codimension two submanifolds of arbitrary manifolds. As the surface  $\Sigma$  is Liouville, then we can carry out the above procedure by performing Liouville connect sums by identifying each of the  $N_j$  and  $N'_j$  as standard neighborhoods of  $\Sigma$  as in the discussion following Definition 1.3 and Section 3.1.  $\square$

**8.5. Kirby diagrams from the proof of Theorem 1.9.** Now we will give an example of a Stein cobordism associated to a branched cover as described in Theorem 1.20. By combining the proof of this theorem with the proof of Theorem 1.9(2), we will be able to give a Kirby diagram description of the cobordism as in [Gom98]. This example should serve as a guide as to how to use the proof of Theorem 1.9(2) to explicitly describe cobordisms associated to the Liouville connect sum in terms of Legendrian surgery. For a similar construction, see [HPK09] where an algorithm is described which produces a contact surgery diagram of a cyclic branched cover, branched over a transverse braid in  $(S^3, \xi_{std})$ .

Throughout this section figures will be drawn in the front projection  $\mathbb{R}^3 \rightarrow \{0\} \times \mathbb{R}^2$  of  $(\mathbb{R}^3, \xi_{std} = \text{Ker}(dz - ydx))$ . Here  $(\mathbb{R}^3, \xi_{std})$  is identified with the complement of a point in  $(S^3, \xi_{std})$ .

Consider the Legendrian graph in Figure 7. Using [Av11, §4] we can draw its ribbon  $\Sigma$  in the front projection. The boundary  $C$  of this ribbon is a Whitehead double of a homologically non-trivial knot in the contact manifold  $(L(2, 1), \xi_{std}) = (S^*S^2, \xi_{can})$  - which can be described by a Legendrian surgery along an unknot with  $tb = -1$ .

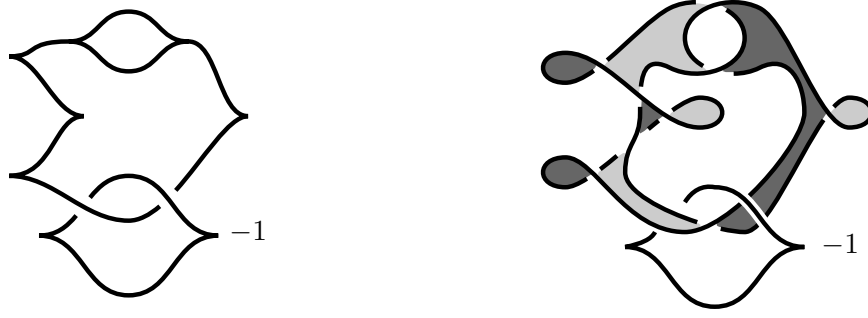


FIGURE 7. On the left is a Legendrian graph in the contact manifold  $(L(2, 1), \xi_{std})$ . The ambient contact manifold is presented as the result of a Legendrian (or, equivalently, a contact  $-1$ -) surgery on a Legendrian unknot with  $tb = -1$ . We will omit the surgery coefficient associated to this unknot in subsequent diagrams. On the right hand side of the figure is the ribbon  $\Sigma$  of the graph. The boundary of  $\Sigma$  is the transverse knot  $T$ .

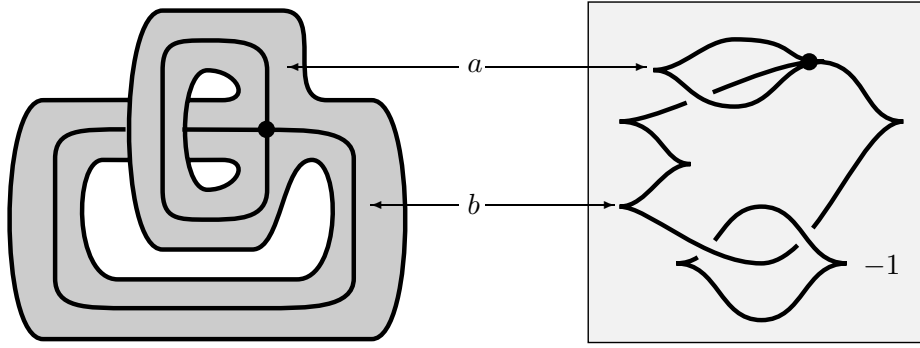


FIGURE 8. A Weinstein handle decomposition of the surface  $\Sigma$  gives rise to an isotropic graph in the ambient contact manifold.

By Theorem 1.20, there is a Stein cobordism from  $\sqcup^q(L(2, 1), \xi_{std})$  to the  $q$ -fold cyclic branched cover  $(S^3, \xi_{std})_{C,q}$ . We will provide a Kirby diagram for this cobordism in the case  $q = 2$  and then describe a completed diagram for the case  $q = 3$ . According to the proof of Theorem 1.20, we can describe  $(S^3, \xi_{std})_{C,2}$  by taking two copies of  $(L(2, 1), \xi_{std})$  each containing a copy of  $\Sigma$  and then perform a Liouville connect sum to the disjoint union of two copies of  $L(2, 1)$  by identifying the copies of  $\Sigma$ .

In our situation, the Liouville surface  $\Sigma$  is a genus 1 surface with a single non-empty boundary component. Therefore  $\Sigma$  admits a Weinstein handle decomposition as a pair of 2-dimensional 1-handles attached to a single disk as depicted in the left-hand side of Figure 8. There, the 0-handle is marked with a black dot which we will call  $p$ . We label the curves of the core disks of the 2-dimensional 1-handles  $a$  and  $b$ . The left-hand side of Figure 8 shows the curves  $a$  and  $b$  embedded in  $(L(2, 1), \xi_{std})$ . Consider  $a$  and  $b$  to be oriented counterclockwise in the figure.

Now we consider two disjoint copies of  $(L(2, 1), \xi_{std})$ , each containing the surface  $\Sigma$ , and so the graph  $a \cup b \cup p$ . We will call one of the surfaces  $\Sigma_1$  and the other  $\Sigma_2$  and fix a diffeomorphism between them induced from the identification of the two copies of  $(L(2, 1), \xi_{std})$ . Similarly we will label the graphs  $a \cup b \cup p$  in each of the copies of  $L(2, 1)$  by  $a_j \cup b_j \cup p_j$ ,  $j = 1, 2$ .

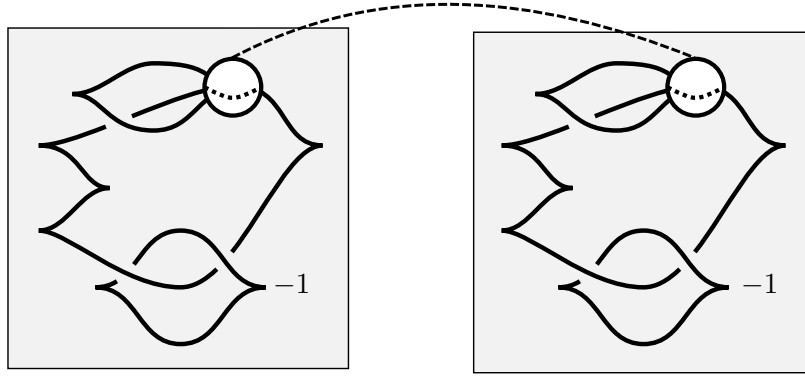


FIGURE 9. A Kirby diagram for the 2-fold cyclic branched cover of  $(L(2, 1), \xi_{std})$  over  $T$ . The boxes represent the fact that the concave end of the cobordism is disconnected. Weinstein 1-handles are represented by spheres connected by dashed lines. The Legendrian knots in the diagram are the attaching loci of the Weinstein 2-handles, and are drawn as thick, black lines. A description of how Legendrian arcs are identified when passing through the 1-handles is described in the text.

The proof of Theorem 1.20 tells us that we can describe the double branched cover of  $(L(2, 1), \xi_{std})$  over  $T$  by Liouville connect summing the two copies of  $L(2, 1)$  along  $\Sigma_1$  and  $\Sigma_2$ , using the identification  $\Sigma_1 \cong \Sigma_2$  described in the previous paragraph. Theorem 1.9 then tells us that this double branched cover can be realized as the convex boundary component of a symplectic cobordism  $(W, \lambda)$  whose concave boundary is  $\sqcup^2(L(2, 1), \xi_{std})$ . The proof of Theorem 1.9(2) described in Section 7.2 provides a handle decomposition of this cobordism as follows:

- (1) Each 2-dimensional 0-handle of the surface  $\Sigma$  gives rise to a 4-dimensional 1-handle in  $(W, \lambda)$ . We attach a 4-dimensional 1-handle to the compact symplectization of  $\sqcup^2(L(2, 1), \xi_{std})$  along 3-dimensional disks centered about the points  $p_1$  and  $p_2$ .
- (2) Each 2-dimensional 1-handle of  $\Sigma$  gives rise to a 4-dimensional 1-handle in  $(W, \lambda)$ . The proof shows that we are to attach one of these two handles along the Legendrian knot  $a_1 \cup (-a_2)$  and another along  $b_1 \cup (-b_1)$ . These knots are indeed closed by identifying  $\partial(a_1)$  with  $\partial(-a_2)$  and  $\partial(b_1)$  with  $\partial(-b_2)$  using the 1-handle attachment along  $p_1$  and  $p_2$ .

Figure 9 shows the completed diagram. Performing the Weinstein handle attachments described above provides a Stein cobordism from  $\sqcup^2(L(2, 1), \xi_{std})$  to  $(L(2, 1), \xi_{std})_{C,2}$ .

Now we will briefly describe the Stein cobordism from  $\sqcup^3(L(2, 1), \xi_{std})$  to the triple branched cover  $(L(2, 1), \xi_{std})_{C,3}$  over the transverse knot  $T$ . This time we start with 3 copies of  $(L(2, 1), \xi_{std})$ . One of the copies contains one copy  $\Sigma_1$  of  $\Sigma$ , another contains two copies  $\Sigma_2$  and  $\Sigma'_2 = \exp(-\epsilon \cdot \partial_z)(\Sigma_2)$  of  $\Sigma$ , and the last contains a single copy  $\Sigma_3$  of  $\Sigma$ . Here,  $\epsilon$  is an arbitrarily small positive constant. The proof of Theorem 1.20 indicates that we can describe the branched cover by performing two Liouville connect sums; the first identifying  $\Sigma_1$  with  $\Sigma_2$ , while the second identifies  $\Sigma'_2$  with  $\Sigma_3$ . Again, by following the proof of Theorem 1.9(2) we obtain a Weinstein handle decomposition of the associated cobordism as in the case of the double-branched cover, described above. The completed diagram is shown in Figure 10.

**8.6. Exact cobordisms which are not Stein.** In this section we construct some high-dimensional Liouville domains which do not admit Stein structures. The examples below serve to illustrate that, in general, the cobordism described in Theorem 1.9 is not Stein.

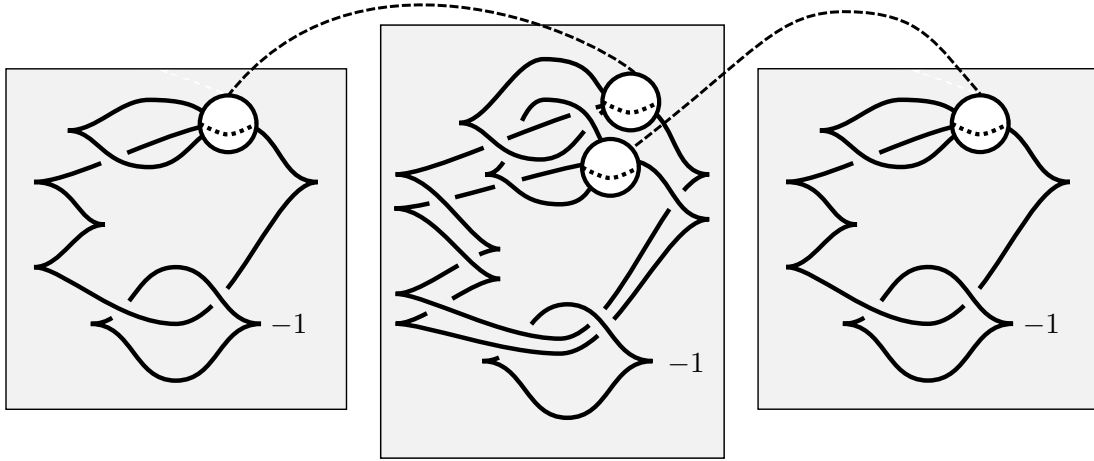


FIGURE 10. A Kirby diagram for the 3-fold cyclic branched cover of  $(L(2,1), \xi_{std})$  branched over  $T$ .

Note that by Theorem 2.11, a connected Stein domain  $(\Sigma, \beta)$  of dimension greater than 2 must have connected boundary. The first examples of connected 4-dimensional Liouville domains whose boundaries are disconnected (and therefore, are not Stein) were discovered by McDuff in [Mc91]. The examples were obtained by modifying the contact form  $-\lambda_{can}$  on the unit cotangent disk bundle  $\mathbb{D}^*S_g$  of a closed, oriented, genus  $g > 1$  surface away from a neighborhood  $N(S_g)$  of its zero section, creating a Liouville 1-form on  $[-1, 1] \times S^*S_g \cong \mathbb{D}^*S_g \setminus \text{Int}(N(S_g))$ . In [Ge94], Geiges generalized this construction, listing a set of conditions associated to a fixed odd-dimensional manifold  $M$  which guarantee the existence of a Liouville 1-form on the product  $[-1, 1] \times M$  and providing examples in the case  $\dim(M) = 5$ . Examples of Liouville 1-forms of manifolds of the form  $[-1, 1] \times M$  – with  $\dim(M)$  being an arbitrary positive odd integer – are described by Massot-Niederkrüger-Wendl in [MNW11, Theorem C].

The cobordism associated to a Liouville connect sum, performed on the convex boundary of a symplectic cobordism  $(W, \omega)$ , either preserves or decreases the number of convex boundary components of  $(W, \omega)$ . To establish that certain cobordisms constructed using Theorem 1.9 are not Stein, we can use singular homology instead of numbers of boundary components.

**Lemma 8.21.** *Suppose that  $(W, \lambda)$  is a connected  $(2n+2)$ -dimensional, Stein-type cobordism with concave boundary  $(M, \xi)$ . Then the inclusion map of  $M$  into  $W$  induces isomorphisms*

$$H_k(W; \mathbb{Z}) \cong H_k(M; \mathbb{Z}) \quad \forall k > n + 1.$$

*In particular, if  $(W, \lambda)$  is a  $(2n+2)$ -dimensional Stein domain, then  $W$  has the homotopy type of an  $(n+1)$ -dimensional CW complex and so  $H_k(W; \mathbb{Z}) = 0$  for all  $k > n + 1$ .*

*Proof.* This follows immediately from Theorem 2.11(4) and the application of a Mayer-Vietoris sequence associated to a Weinstein handle attachment.  $\square$

**Theorem 8.22.** *Let  $(M, \xi)$  be a  $(2n+1)$ -dimensional contact manifold where  $n > 1$  and let  $T$  be a closed  $(2n-1)$ -dimensional manifold. Suppose that  $[-1, 1] \times T$  has a Liouville 1-form  $\beta$  and that there are disjoint Liouville embeddings  $i_1, i_2 : ([-1, 1] \times T, \beta) \rightarrow (M, \xi)$ . Suppose further that*

$$i_1[T] = i_2[T] \quad \text{in} \quad H_{2n-1}(M, \mathbb{Z}).$$

*Then the exact cobordism  $(W, \lambda)$  of Theorem 1.9 associated to the Liouville connect sum of  $(M, \xi)$  along the  $i_j([-1, 1] \times T)$  ( $j = 1, 2$ ) is not Stein.*

*Proof.* Again, apply a Mayer-Vietoris sequence to the handle-attachment pair  $([1/2, 1] \times M, H_{[-1, 1] \times T})$ . It follows that  $H_{2n}(W; \mathbb{Z}) \cong H_{2n}(M; \mathbb{Z}) \oplus \mathbb{Z}$ , completing the proof by Lemma 8.21.

The extra “ $\mathbb{Z}$ ” factor in  $H_{2n}(W; \mathbb{Z})$  can be explicitly described as follows: Let  $X$  be an oriented  $(2n)$ -dimensional cobordism from  $T$  to itself such that there is a map  $i_X : X \rightarrow M$  with  $i_X(\partial X) = i_1(T) \cup i_2(T)$ . Let  $Y = [-1, 1] \times T$  and let  $i_Y : Y \rightarrow H_{[-1, 1] \times T} = [-1, 1] \times N([-1, 1] \times T)$  be the map  $(\theta, x) \mapsto (\theta, 0, 0, x)$ . Here we are using the coordinates on a standard neighborhood as described in the proof of Theorem 1.9. Then  $i_X[X] + i_Y[Y] \in H_{2n}(W, \mathbb{Z})$  generates the desired homology class.  $\square$

Now we will provide some concrete examples.

**Example 8.23.** Let  $(\Sigma, \beta)$  be a  $(2n)$ -dimensional Liouville domain for which  $\Sigma = [-1, 1] \times T$  for some closed, smooth  $(2n - 1)$ -dimensional manifold  $T$ . Here we require that  $n > 1$ . As pointed out in the discussion following the statement of Theorem 1.16,  $(\mathbb{D}^2 \times \Sigma, \lambda_{std} + \beta)$  is a Liouville domain whose boundary is a contact manifold admitting an open book determined by the pair  $((\Sigma, \beta), \text{id}_\Sigma)$ . For each  $\theta \in S^1 = [0, 2\pi]/\sim$  denote by  $\Sigma_\theta$  the page of this open book corresponding to  $\theta$ , with a collar neighborhood of its boundary removed. Then each  $\Sigma_\theta$  is a Liouville hypersurface in the contact manifold  $\partial(\mathbb{D}^2 \times \Sigma, \lambda_{std} + \beta)$ .

Choose an even natural number  $2g$  and let  $\sigma \in S_{2g}$  be a permutation which fixes no element of the set  $\{1, \dots, 2g\}$ . Perform  $g$ -Liouville connect sums along  $\partial(\mathbb{D}^2 \times \Sigma, \lambda_{std} + \beta)$ , identifying each  $\Sigma_{\pi^j/k}$  with  $\Sigma_{\pi^{\sigma(j)}/k}$ . Using the handle attachment described in the proof of Theorem 1.9 we can view this new contact manifold as the boundary of a Liouville domain whose underlying manifold is diffeomorphic to  $H_g \times T$  for a genus- $g$  handlebody  $H_g$ .

## 9. CONTACT $(1/k)$ -SURGERIES ON CONTACT MANIFOLDS OF ARBITRARY DIMENSION

The purpose of this section is to define contact  $(1/k)$ -surgery and describe some of its basic properties as stated in the introduction. For the purpose of motivation, we begin by giving a brief overview of contact surgery on contact 3-manifolds as defined in [DG01].

**9.1. The case  $\dim(M)=3$ .** Suppose that  $(M, \xi)$  is a contact 3-manifold containing a Legendrian knot  $L$  and let  $k$  be any integer. Then  $L$  admits a tubular neighborhood  $N(L)$  in  $(M, \xi)$  of the form  $N(L) = [-\epsilon, \epsilon] \times \mathbb{D}^*S^1 = [-\epsilon, \epsilon] \times [-1, 1] \times S^1$  on which  $\xi = \text{Ker}(dz - \lambda_{can})$ . To perform *contact  $(1/k)$ -surgery* on  $L \subset (M, \xi)$ , remove  $N(L)$  from  $(M, \xi)$  and glue it back using a map which is boundary-relative isotopic to  $-k$  Dehn twists along  $\{\epsilon\} \times \mathbb{D}^*S^1$  and isotopic to the identity on the remainder of the boundary of  $N(L)$ . The boundary relative isotopy class of the Dehn twists may be chosen in such a way that the surgered manifold naturally admits a contact structure, which depends only on  $(M, \xi)$ , the Legendrian isotopy class of  $L$  in  $(M, \xi)$ , and the integer  $k$ . See [DG01, Proposition 7].

**Remark 9.1.** *In the language of this paper, we can perform contact  $(1/k)$ -surgery along  $L \subset (M, \xi)$  as follows: Remove a standard neighborhood  $\mathcal{N}(\Sigma)$  of a ribbon  $\Sigma$  of  $L$ , and reattach  $\partial\mathcal{N}(\Sigma)$  to  $\partial(M \setminus \text{Int}(\mathcal{N}(\Sigma)))$  using the convex gluing instructions  $(\tau^{-k}, \text{id}_\Sigma)$ . Here  $\tau$  denotes a positive Dehn twist along  $\Sigma$  which can be identified with an annulus.*

**Theorem 9.2.** *Let  $(M, \xi)$  be a connected contact 3-manifold.*

- (1) *Performing a contact  $(-1)$ -surgery along any Legendrian knot  $L \subset (M, \xi)$  gives the same contact manifold as the one obtained by performing a 4-dimensional Weinstein 2-handle attachment along  $L$ .*
- (2) *For any Legendrian knot  $L \subset (M, \xi)$ , the contact manifold described by performing a contact  $(1/p)$ -surgery on  $L$ , followed by a  $(1/q)$ -surgery on a push-off of  $L$  is equivalent to the contact manifold described by performing a  $(1/(p + q))$ -surgery on  $L \subset (M, \xi)$ .*



FIGURE 11. A Kirby diagram depicting the statement of Theorem 9.2(3). We emphasize that the diagram on the right only represents a 3-manifold, not a Stein 4-manifold.

- (3) *Performing a contact  $(+1)$ -surgery on a standard Legendrian unknot in  $(M, \xi)$  produces the contact connect sum of  $(M, \xi)$  with  $(S^1 \times S^2, \xi_{std}) = \partial(\mathbb{D}^*S^1 \times \mathbb{D}^2, -\lambda_{can} + \lambda_{std})$ . See Figure 11.*
- (4) *Performing a contact  $(1/2)$ -surgery on a standard Legendrian unknot in  $(M, \xi)$  yields  $M$  equipped with an overtwisted contact structure.*
- (5) *Performing a contact  $(+1)$ -surgery on a stabilized Legendrian knot in  $(M, \xi)$  produces an overtwisted contact manifold.*
- (6) *There is a Legendrian link  $L = L^+ \cup L^-$  in  $(S^3, \xi_{std})$  such that performing  $(+1)$ -surgery along the components of  $L^+$  and  $(-1)$ -surgery along the components of  $L^-$  yields  $(M, \xi)$ .*

For proofs of the above statements, we refer the reader to the exposition [OS04, §11.2] and the references therein. An alternate proof of item (6) can be found in [Av11]. Statements (1-4) in the above theorem can also be viewed as special cases of Theorem 9.15, below.

**9.2. Generalized Dehn twists.** As the reader may suspect from Remark 9.1, the essential ingredient in our definition of contact  $(1/k)$ -surgery is the *generalized Dehn twist*, first discovered in the context of symplectic geometry by Arnol'd in [Ar95] and further popularized in the work of Seidel [S97, S99].

Identify the cotangent bundle of the  $n$ -sphere with the set

$$T^*S^n = \{(u, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \|u\| = 1, \langle u, v \rangle = 0\}.$$

Here  $\langle *, * \rangle$  denotes the standard inner product on  $\mathbb{R}^{n+1}$ . We consider  $T^*S^n$  as a symplectic manifold with the canonical symplectic form  $-d\lambda_{can} = \sum_1^{n+1} du_i \wedge dv_i$ . In this model situation we can write  $-\lambda_{can} = -\sum_1^{n+1} v_i du_i$ . Fix an arbitrarily small positive constant  $\epsilon < 1$ , and let  $f : [0, \infty) \rightarrow \mathbb{R}$  be a function such that

- (1)  $f(0) = \pi$  and all derivatives vanish on a neighborhood of 0,
- (2)  $f$  is non-decreasing, and
- (3)  $f(x) = 2\pi$  for all  $x \geq \epsilon$ .

Now define the diffeomorphism  $\tau_n : T^*S^n \rightarrow T^*S^n$  determined by the formula

$$\tau_n(u, v) = (\cos \circ f(\|v\|) \cdot u + \sin \circ f(\|v\|) \cdot \frac{v}{\|v\|}, -\|v\| \sin \circ f(\|v\|) \cdot u + \cos \circ f(\|v\|) \cdot v).$$

The diffeomorphism  $\tau_n$  coincides with the identity mapping on a tubular neighborhood of  $\partial\mathbb{D}^*S^n \subset T^*S^n$  by our assumption that  $\epsilon < 1$ . Hence we will view  $\tau_n$  as an element of  $\text{Diff}^+(\mathbb{D}^*S^n, \partial\mathbb{D}^*S^n)$ .

**Theorem 9.3.** *The diffeomorphism  $\tau_n$  preserves  $-d\lambda_{can}$  and its isotopy class in  $\text{Symp}((\mathbb{D}^*S^n, -d\lambda_{can}), \partial\mathbb{D}^*S^n)$  is independent of the constant  $\epsilon < 1$  and the function  $f$ .*

A proof may be found in [S97, S99].

**Definition 9.4.** *We call any symplectomorphism which is boundary-relative symplectically isotopic to  $\tau_n \in \text{Symp}((\mathbb{D}_\epsilon^*, -d\lambda_{can}), \partial\mathbb{D}_\epsilon^*S^n)$  a generalized Dehn twist.*

It is easy to see that the mapping  $\tau_1$  coincides with the usual notion of a Dehn twist on an annulus.



**Remark 9.5.** *The reader may note that our choices of orientation on  $T^*S^n$  and conditions defining the function  $f$  above are the opposite of those often appearing in the literature. However, the end result is the same. See [S99, Remark 6.4].*

**Example 9.6** ([Ar95]). Consider the function  $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$  given by

$$(z_1, \dots, z_{n+1}) \mapsto \sum_{j=1}^{n+1} z_j^2.$$

This function induces an open book decomposition of  $S^{2n+1} = \partial \mathbb{D}^{2n+2}$  whose binding is  $f^{-1}(0) \cap S^{2n+1}$ . In this case, each page is diffeomorphic to  $\mathbb{D}^*S^n$  and the monodromy is given by a generalized Dehn twist. Moreover, this open book is compatible with the standard contact structure  $\xi_{std}$  on  $S^{2n+1}$ .

**9.3. Contact  $(1/k)$ -surgery.** We give two equivalent definitions of contact  $(1/k)$ -surgery. The first will make it clear that our definition extends the 3-dimensional one described above, while the second will make it easier to prove some of its basic properties. Afterwards we briefly discuss a subtlety in this definition, which is irrelevant when performing surgery on 3-dimensional contact manifolds.

**9.3.1. First definition.** Let  $(M, \xi)$  be any  $(2n+1)$ -dimensional contact manifold containing a Legendrian sphere  $L$ . The Weinstein neighborhood theorem for Legendrian submanifolds asserts that for any contact form  $\alpha$  for  $(M, \xi)$ , we can find a ribbon  $\Sigma = \mathbb{D}^*L$  for which  $\alpha|_{\mathbb{D}^*L} = -\lambda_{can}$ . Consider a Liouville embedding  $I : (\mathbb{D}^*S^n, -\lambda_{can}) \rightarrow (M, \xi)$  whose image is the ribbon  $\Sigma$  of  $L$ .

**Definition 9.7.** *To perform contact  $(1/k)$ -surgery on  $L$  with parameter  $I$ , remove a standard neighborhood  $\mathcal{N}(\Sigma)$  of  $\Sigma$  from  $(M, \xi)$  and then reattach  $\partial \mathcal{N}(\Sigma)$  to  $\partial(M \setminus \text{Int}(\mathcal{N}(\Sigma)))$  using the convex gluing instructions  $(\tau^{-k}, id_\Sigma)$ .*

**9.3.2. Second definition.**

**Definition 9.8.** *For each natural number  $n > 0$  define  $(M, \xi)_{n,k}$  to be the  $(2n+1)$ -dimensional contact manifold determined by the open book for the pair  $((\mathbb{D}^*S^n, -\lambda_{std}), \tau_n^k)$ , where  $\tau_n^k$  denotes the  $k$ -fold iterate of  $\tau_n$ .*

**Example 9.9.** The smooth manifold underlying  $(M, \xi)_{n,-1}$  is  $S^{2n+1}$ , although its contact structure is not  $\xi_{std}$ . As shown in [BK10],  $(M, \xi)_{n,-1}$  is not symplectically fillable for any  $n$ . This fact is well known in the case  $n = 1$ , as  $(M, \xi)_{1,-1}$  is overtwisted. For a more explicit description of  $(M, \xi)_{n,-1}$ , see [NP10, Example 5].

By Example 9.6  $(M, \xi)_{n,1}$  is diffeomorphic to the standard contact sphere  $(S^{2n+1}, \xi_{std})$ . As pointed out in the discussion following the statement of Theorem 1.16,  $(M, \xi)_{n,0}$  can be realized as the boundary of the Liouville domain  $(\mathbb{D}^2 \times \mathbb{D}^*S^n, \lambda_{std} - \lambda_{can})$ .

We claim that the contact manifold  $(M, \xi)_{n,2}$  coincides with the canonical contact structure on the unit cotangent bundle  $S^*S^{n+1}$ . By Examples 2.10, we see that  $(S^*S^{n+1}, \xi_{can})$  can be realized by performing a Weinstein handle attachment along a Legendrian sphere in  $(S^{2n+1}, \xi_{std})$ . According to Example 9.6, this Legendrian sphere can be realized as the zero section of  $\mathbb{D}^*S^n$  which we identify with one of the pages of the open book used to describe  $(M, \xi)_{n,1} = (S^{2n+1}, \xi_{std})$ . According to Example 1.10, the surgered contact manifold  $(S^*S^{n+1}, \xi_{can})$  can be described by performing a Liouville connect sum on two disjoint copies of  $(M, \xi)_{n,1}$ , by identifying a page of one copy with a page of the other copy. According to the proof of Theorem 1.16, the resulting contact manifold is  $(M, \xi)_{n,2}$ . Thus  $(M, \xi)_{n,2} = (S^*S^{n+1}, \xi_{can})$ .

Again, let  $(M, \xi)$  be any  $(2n+1)$ -dimensional contact manifold containing a Legendrian sphere  $L$  and consider a Liouville embedding  $I : (\mathbb{D}^*S^n, -\lambda_{can}) \rightarrow (M, \xi)$  whose image is the ribbon  $\Sigma$  of  $L$ . We have a fixed identification of  $\mathbb{D}^*S^n$  with a page of the open book  $(M, \xi)_{n,k}$  for each  $k$  as this manifold is defined constructively.

**Definition 9.10.** Define the contact manifold  $(M, \xi)_{(L, I, k)}$  as the Liouville connect sum of  $(M, \xi) \sqcup (M, \xi)_{n, -k}$  using the Liouville embedding  $I$ . We say that  $(M, \xi)_{(L, I, k)}$  is obtained by contact  $(1/k)$ -surgery on  $L$  with parameter  $I$ .

This definition, together with Theorem 1.9 allows us to associate a Stein cobordism to a contact  $(1/k)$ -surgery. In Figure 12 we provide a Kirby diagram for one such cobordism.

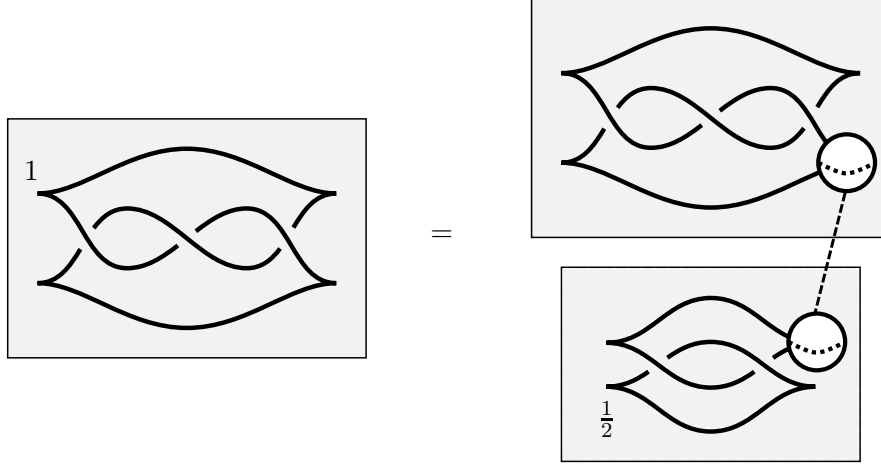


FIGURE 12. On the right we have a Stein cobordism whose concave boundary is a disjoint union of  $(S^3, \xi_{std})$  with the overtwisted 3-sphere  $(M, \xi)_{1, -1}$ . The overtwisted sphere is given as the result of a  $(1/2)$ -surgery on the standard Legendrian unknot in  $(S^3, \xi_{std})$ . The convex end of this cobordism is equivalent is equal to the contact manifold on the right, described by a  $(+1)$ -surgery on a right-handed Legendrian trefoil in  $(S^3, \xi_{std})$ . The cobordism is decomposed into two Weinstein handle attachments; a 4-dimensional 1-handle attachment labeled by the 3-disks attached by a dotted line, and a 4-dimensional 2-handle determined by the unlabeled Legendrian knot which passes twice through the 1-handle. This handle decomposition is given by applying the reasoning of Section 8.5 to the decomposition of  $(\mathbb{D}^*S^1, -\lambda_{can})$  into a 2-dimensional 0-handle together with a single 2-dimensional 1-handle.

**Proposition 9.11.** Definitions 9.7 and 9.10 are equivalent.

*Proof.* As in Lemma 8.2, we can present  $(M, \xi)_{n, k}$  as two copies  $N_1$  and  $N_2$  of a standard neighborhood of  $(\mathbb{D}^*S^n, -\lambda_{can})$  whose boundaries are identified via the convex gluing instructions  $(\tau_n^k, \text{id}_{\mathbb{D}^*S^n})$ . Hence we can perform contact  $(1/k)$ -surgery as described in Definition 9.10 by removing  $N_2$  from  $(M, \xi)_{n, k}$ , removing  $\mathcal{N}(\Sigma)$  from  $(M, \xi)$  and performing a convex gluing. This is clearly equivalent to Definition 9.7.  $\square$

**9.3.3. Dependence on the parameter  $I$ .** In general the contact manifold  $(M, \xi)_{(L, I, k)}$  depends a priori on the parametrization  $I$  – not just the image  $L$  of  $I$ . For example, by combining Examples 1.10 and 2.10, we see that different parametrization of a Legendrian sphere in  $(S^{2n+1}, \xi_{std})$  can produce contact manifolds which may be inequivalent. There are however certain cases in which we can guarantee that  $(M, \xi)_{(L, I, k)}$  is independent of  $I$ .

**Proposition 9.12.** Suppose that  $H : S^n \times [0, 1] \rightarrow M$  is an isotopy of Legendrian spheres in  $(M, \xi)$ , i.e. for each  $t \in [0, 1]$   $H(*, t) : S^n \rightarrow M$  yields an embedded Legendrian sphere. Writing  $H(*, 0) = I$ ,  $H(*, 1) =$

$I'$ ,  $I(S^n) = L$ , and  $I'(S^n) = L'$  we have that  $(M, \xi)_{(L, I, k)}$  is contact-diffeomorphic to  $(M, \xi)_{(L', I', k)}$ . Moreover, if  $\text{Diff}^+(S^n)$  is path connected, then  $(M, \xi)_{(L, I, k)}$  is independent of the parametrization  $I$ .

The first statement follows from the fact that we can write  $H(*, t) = \phi_t \circ H(*, 0)$  for an isotopy  $\phi_t$  of  $M$  which preserves  $\xi$ . See [Et05, Theorem 2.12]. The second statement is essentially [S99, Lemma 6.2]. The hypothesis of the second statement is known to hold true for  $n = 1, 2, 3, 4, 5, 11, 60$  and is known to not hold true for any other values of  $n \leq 63$ . See [Mi11] and the references listed therein.

**9.4. Basic properties.** Now we outline some basic properties of contact  $(1/k)$ -surgery, showing that many of the results of Theorem 9.2 continue to hold in high dimensions. In order to state our results, we must first define the *standard Legendrian sphere* and *Legendrian push-offs*.

**Definition 9.13.** The standard Legendrian sphere in  $(S^{2n+1}, \xi_{std})$ , denoted  $L_{std}$ , is given by  $S^{2n+1} \cap \text{Span}(x_1, \dots, x_{n+1})$  where we consider  $(S^{2n+1}, \xi_{std}) = \partial(\mathbb{D}^{2n+2}, \lambda_{std})$ . Let  $(M, \xi)$  be a  $(2n+1)$ -dimensional contact manifold and identify  $(M, \xi)$  with the contact connect sum of  $(M, \xi)$  and  $(S^{2n+1}, \xi_{std})$ , where the connect sum is performed outside of a tubular neighborhood of  $L_{std} \subset S^{2n+1}$ . In this way, we view  $L_{std}$  as a Legendrian sphere in  $(M, \xi) = (M, \xi) \# (S^{2n+1}, \xi_{std})$ . We say that a Legendrian sphere  $L$  in  $(M, \xi)$  is a standard Legendrian sphere if it is Legendrian isotopic to  $L_{std} \subset (M, \xi)$ .

According to the above definition, a standard Legendrian sphere in a contact 3-manifold is a Legendrian unknot with Thurston-Bennequin number equal to  $-1$ . The sphere  $L_{std}$  has a canonical parametrization given by its identification with the unit  $n$ -sphere in  $\text{Span}(x_1, \dots, x_{n+1})$ .

**Definition 9.14.** Let  $L \subset (M, \xi)$  be a Legendrian submanifold and identify a tubular neighborhood  $N(L)$  of  $L$  with  $N(L) = [-\epsilon, \epsilon] \times \mathbb{D}^*L$ . We say that a Legendrian submanifold of  $(M \setminus L, \xi)$  is a push-off of  $L$  if it is Legendrian isotopic to  $\{\epsilon\} \times L \subset N(L) \setminus L$ .

This notion of push-off clearly extends the usual definition of a push-off of a Legendrian knot in a contact 3-manifold. A parametrization  $I : S^n \rightarrow L$  of a Legendrian sphere in  $(M, \xi)$  gives rise to a canonical parametrization of a push-off by  $(\epsilon, I) : S^n \rightarrow N(L)$ .

**Theorem 9.15.** Let  $(M, \xi)$  be a connected  $(2n+1)$ -dimensional contact manifold.

- (1) Performing a contact  $(-1)$ -surgery along any Legendrian sphere  $L \subset (M, \xi)$  gives the same contact manifold as the one obtained by performing a  $(2n+2)$ -dimensional Weinstein  $(n+1)$ -handle attachment along  $K$ .
- (2) For any Legendrian sphere  $L \subset (M, \xi)$ , the contact manifold described by performing a contact  $(1/p)$ -surgery on  $L$  with parameter, followed by a  $(1/q)$ -surgery on a push-off of  $L$  (with its natural parametrization) is equivalent to the contact manifold described by performing a  $(1/(p+q))$ -surgery on  $K \subset (M, \xi)$ .
- (3) Performing a contact  $(+1)$ -surgery on a standard Legendrian sphere in  $(M, \xi)$  with its natural parametrization produces the contact connect sum of  $(M, \xi)$  with  $(S^n \times S^{n+1}, \xi_{std}) := \partial(\mathbb{D}^*S^n \times \mathbb{D}^2, -\lambda_{can} + \lambda_{std})$ .
- (4) Performing a contact  $(1/2)$ -surgery on a standard Legendrian sphere in  $(M, \xi)$  yields  $M$  equipped with an algebraically overtwisted (and so not symplectically fillable) contact structure.

*Proof.* **1.** This is a combination of Examples 1.10 and 9.6.

**2.** Suppose that we perform a  $(1/p)$ -surgery on  $L \subset (M, \xi)$  using a Liouville embedding  $I : (\mathbb{D}^*S^n, -\lambda_{can}) \rightarrow (M, \xi)$ . Write  $\mathcal{N}(\Sigma)$  for a standard neighborhood of the image  $\Sigma$  of  $I$ . Legendrian isotop the push-off of  $L$  into the interior of  $\mathcal{N}(\Sigma)$  – now considered as a subset of  $(M, \xi)_{(L, I, p)}$  – in the obvious fashion so that it is identified with the zero-section of  $\mathbb{D}^*S^n$  which we considered to be a page of  $(M, \xi)_{n, -p}$  by alternately thinking of  $\mathcal{N}(\Sigma)$  as a subset of  $(M, \xi)_{n, -p}$ . Performing  $(1/q)$ -surgery on this sphere in  $(M, \xi)_{n, p}$  amounts

to adding  $-k$  Dehn twists to the monodromy of the open book determining  $(M, \xi)_{n,-p}$  and so results in  $(M, \xi)_{n,-p-q}$ . Therefore the end result is a Liouville connect sum of  $(M, \xi)$  and  $(M, \xi)_{n,-p-q}$  along  $\Sigma$  using the parametrization  $I$ .

**3.** A contact manifold obtained by surgery as in the statement of item (3) above is a contact connect sum of  $(M, \xi)$  with  $(M, \xi)_{n,0}$ . As pointed out in the discussion following the statement of Theorem 1.16,  $(M, \xi)_{n,0}$  is the boundary of the Liouville domain obtained by rounding the corners of  $(\mathbb{D}^*S^n \times \mathbb{D}^2, -\lambda_{can} + \lambda_{std})$ .

**4.** A contact manifold obtained by surgery as in the statement of item (4) above is the same as is given by the contact connect sum of  $(M, \xi)$  with  $(M, \xi)_{n,-1}$ . Therefore, this assertion follows from [BK10].  $\square$

In [EES05, §4.1] a notion of “stabilized Legendrian sphere” is described, in analogy with the usual notion of a stabilized Legendrian knot in a contact 3-manifold. Certain stabilized spheres, called *loose Legendrian spheres* are classified (up to Legendrian isotopy) by homotopy theoretic data [Mu12]. Based on Theorem 9.2(5) the fact that these spheres have trivial holomorphic curve invariants – see [EES05, Proposition 4.8] and [Mu12, §8] – we present the following as a conjectured analogue of Theorem 9.2(5).

**Conjecture 9.16.** *Let  $L \subset (M, \xi)$  be a loose Legendrian sphere in a contact manifold of dimension greater than three. A contact manifold obtained by performing a contact  $(+1)$ -surgery along  $L$  is “overtwisted”.*

Here “overtwistedness” of a contact manifold  $(M, \xi)$  of dimension greater than three can be taken to mean – or at least imply – any of the following conditions:

- (1)  $(M, \xi)$  does not admit a weak symplectic filling.
- (2) Every contact form on  $(M, \xi)$  has a contractible Reeb orbit.
- (3) The contact homology of  $(M, \xi)$  is zero for any choice of coefficient system.
- (4)  $(M, \xi)$  contains a plastikstufe as described in [N06].
- (5) The contact structure  $\xi$  is determined by qualitative and homotopical data as in [El89].

## 10. APPLICATIONS TO THE STUDY OF DEHN TWISTS

For our final application of the Liouville connect sum we use contact  $(1/k)$ -surgery, as described in the previous section, to study squares of generalized Dehn twists. We begin by carrying out the proof of Theorem 1.21.

### 10.1. Squares of smooth Dehn twists.

**Lemma 10.1.** *Suppose that  $n \neq 2, 6$ . Then  $\tau_n^2$  is not isotopic to the identity mapping in  $\text{Diff}^+(\mathbb{D}^*S^n, \partial\mathbb{D}^*S^n)$ .*

*Proof.* If  $\tau_n^2$  is isotopic to the identity in  $\text{Diff}^+(\mathbb{D}^*S^n, \partial\mathbb{D}^*S^n)$ , then the smooth manifolds underlying  $(M, \xi)_{n,0}$  and  $(M, \xi)_{n,2}$  are diffeomorphic. Therefore, according to Example 9.9 it suffices to show that the unit cotangent bundle of  $S^{n+1}$  is not diffeomorphic to  $S^n \times S^{n+1}$  for  $n \neq 0, 2, 6$ . We observe that these spaces are not homotopy equivalent.

If  $n$  is odd, then  $H_n(S^*S^{n+1}; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$  – as can be computed using a Gysin sequence from the fact that  $\chi(S^{n+1}) = 2$  – while  $H_*(S^n \times S^{n+1}, \mathbb{Z})$  has no torsion. In the event that  $n$  is even, then  $S^*S^{n+1}$  is homotopy equivalent to  $S^n \times S^{n+1}$  if and only if  $n+1 = 1, 3, 7$  as can be seen by combining results of Adams [Ad58, Theorem 1(b)] and James-Whitehead [JW54, Theorem 1.12]. Specifically, [Ad58, Theorem 1 (b)] asserts that there is a map  $S^{2n-1} \rightarrow S^n$  with Hopf invariant equal to one if and only if  $n = 1, 2, 4, 8$  while [JW54, Theorem 1.12] asserts that  $S^*S^{n+1}$  is homotopy equivalent to  $S^n \times S^{n+1}$  if and only if there is an element in  $\pi_{2n-1}(S^n)$  with Hopf invariant equal to one.  $\square$

**Lemma 10.2.** *For  $n = 2, 6$  the square of the generalized Dehn twist  $\tau_n^2$  is isotopic to the identity mapping in  $\text{Diff}^+(\mathbb{D}^*S^n, \partial\mathbb{D}^*S^n)$ .*

*Proof.* Our proof is lifted from [S99, Lemma 6.3] where the case  $n = 2$  is established. The mechanism underlying the proof of [S99, Lemma 6.3] is the existence of an (almost) complex structure on  $S^2$ . Our only contribution is the observation that  $S^6$  also admits an almost complex structure as is determined by a cross product on the imaginary octonians. Thus we suppose that  $S^n$  is a sphere equipped with an almost complex structure  $J$ .

Let  $u \in S^n$ . Then  $J_u : T_u S^n \rightarrow T_u S^n$  determines an element  $j_u$  of the Lie algebra  $\mathfrak{so}(n+1)$  of  $SO(n+1)$  as follows. Using the standard metric on  $TS$  provided by the natural inclusion of  $S^n$  into  $\mathbb{R}^{n+1}$ , identify the  $(n-1)$ -sphere of unit length vectors in  $T_u S^n$  with the  $(n-1)$ -sphere of points in  $S^n$  which are orthogonal to  $u$  when considered as vectors in  $\mathbb{R}^{n+1}$ . In this way we can see that  $J_u$  generates a circle subgroup of the subgroup of transformations in  $SO(n+1)$  which fix the point  $u$ . Indeed, for each  $\theta \in S^1 = [0, 2\pi]/\sim$  we can consider that map  $e^{\theta \cdot J_u} : T_u S^n \rightarrow T_u S^n$  as a map  $S^n \rightarrow S^n$  fixing  $u$ . Denote by  $j_u \in \mathfrak{so}(n)$  the infinitesimal generator of this action.

Similarly, if  $v \in T_u^* S^n$  is a non-zero cotangent vector then there is an associated vector  $v_u \in \mathfrak{so}(n+1)$ . Denote by  $v^*$  the associated dual vector in  $T_u S^n$ , which we will consider as a vector in  $\mathbb{R}^{n+1}$ . Define  $v_u$  to be the infinitesimal generator of the  $SO(n+1)$ -circle action on  $S^n$  which rotates the oriented plane  $\text{Span}(u, \frac{1}{\|v^*\|} v^*)$  counterclockwise and fixes the orthogonal complement of this plane.

In the notation of Section 9.2, we can use the above definitions to express the Dehn twist  $\tau_n$  as

$$\tau_n(u, v) = (e^{f(\|v\|) \cdot v_u} u, e^{f(\|v\|) \cdot v_u} v)$$

for each pair  $(u, v)$  satisfying  $v \neq 0$ . Here  $f$  is the function described in Section 9.2. For points of the form  $(u, 0) \in \mathbb{D}^* S^n$ ,  $\tau_n(u, 0) = (-u, 0)$ . Using the above formula we can write

$$\tau_n^2(u, v) = (e^{2f(\|v\|) \cdot v_u} u, e^{2f(\|v\|) \cdot v_u} v)$$

for pairs  $(u, v) \in \mathbb{D}^* S^n$  satisfying  $v \neq 0$ , and  $\tau_n^2(u, 0) = (u, 0)$  for each  $(u, 0) \in S^n \subset \mathbb{D}^* S^n$ .

Consider the  $[0, 1]$ -family of  $SO(n+1)$ -circle actions on  $\mathbb{D}^* S^n$  given by the formula

$$\Phi_t(u, v) = (e^{(2f(\|v\|) \cdot (1-t)j_u + tv_u)} u, e^{2f(\|v\|) \cdot (1-t)j_u + tv_u} v)$$

for  $v \neq 0$  and  $\Phi_t(u, 0) = (u, 0)$  for all  $t \in [0, 1]$ . Note that  $\Phi_1 = \tau_n^2$ , so that  $\Phi_t$  provides an isotopy from  $\tau_n^2$  to the diffeomorphism

$$\Phi_0(u, v) = (u, e^{2f(\|v\|)j_u} v)$$

in such a way that  $\Phi_t$  restricts to the identity mapping along the zero-section and boundary of  $\mathbb{D}^* S^n$  for all  $t \in [0, 1]$ . To complete the proof, consider the isotopy

$$\Psi_t(u, v) = (u, e^{t2f(\|v\|)j_u} v)$$

which interpolates between  $\Phi_0$  and the identity mapping in  $\text{Diff}^+(\mathbb{D}^* S^n, \partial \mathbb{D}^* S^n)$ .  $\square$

This concludes the proof of Theorem 1.21. The above results can be combined to prove the following:

**Corollary 10.3.** *The only spheres which admit almost complex structures are  $S^2$  and  $S^6$ .*

For if any other sphere  $S^n$  had an almost complex structure, then the proof of Lemma 10.2 would imply that  $\tau_n^2$  is isotopic to the identity in  $\text{Diff}^+(\mathbb{D}^* S^n, \partial \mathbb{D}^* S^n)$ , contradicting Lemma 10.1.

**10.2. Squares of symplectic Dehn twists and exotic contact spheres.** As an application of our definition of contact  $(1/k)$ -surgery we establish the existence of exotic contact structures on the spheres  $S^5$  and  $S^{13}$ . Throughout this section, unless stated otherwise, we use  $n$  to denote either 2 or 6.

Let  $\Phi : S^n \rightarrow S^{2n+1}$  be a Legendrian embedding of a  $n$ -dimensional sphere into the standard contact  $(2n+1)$ -sphere  $(S^{2n+1}, \xi_{std})$ . Denote the image of this embedding by  $L$ , and let  $2m$  be a non-zero, even

integer. An infinite family of Legendrian spheres in  $S^{2n+1}$  (for arbitrary  $n$ , not necessarily equal to 2 or 6) which are pairwise not Legendrian isotopic can be found in [EES05, §4.1].

**Definition 10.4.** Define the contact manifold  $(S^{2n+1}, \xi_{std})_{L, \Phi, 2m}$  to be contact manifold obtained by performing contact  $(1/2m)$ -surgery on the Legendrian sphere  $L$  in  $(S^{2n+1}, \xi_{std})$  with parameter  $\Phi$ .

We observe that when  $m < 0$  and the Legendrian sphere  $L \subset (S^{2n+1}, \xi_{std})$  is the standard Legendrian sphere  $L = L_{std}$  described in Definition 9.13 – with its standard embedding – then the contact manifold  $(S^{2n+1}, \xi_{std})_{L, \Phi, 2m}$  is one of the Brieskorn contact manifolds studied in [KN05] and [U99]. Indeed, in this case  $L$  is the core sphere of a page of the open book of  $(S^{2n+1}, \xi_{std})$  whose page is  $\mathbb{D}^*S^{n+1}$  and whose monodromy is a single Dehn twist. Thus  $(S^{2n+1}, \xi_{std})_{L, \Phi, 2m} = (M, \xi)_{n, 1-2m}$  yielding the open books described in [KN05].

**Theorem 10.5.** The smooth manifold underlying  $(S^{2n+1}, \xi_{std})_{L, \Phi, 2m}$  is  $S^{2n+1}$ . However, this contact manifold is not contact-diffeomorphic to  $(S^{2n+1}, \xi_{std})$ .

Our proof of Theorem 10.5 is a simple application of the following theorem of Eliashberg- Floer-Gromov-McDuff [El91, Mc91].

**Theorem 10.6.** Let  $n \geq 1$  be an arbitrary natural number and suppose that  $(W, \omega)$  is a symplectic filling of  $(S^{2n+1}, \xi_{std})$  for which  $\omega$  integrates to zero over every embedded 2-sphere in  $W$ . Then  $W$  is diffeomorphic to the  $(2n+2)$ -dimensional disk,  $\mathbb{D}^{2n+2}$ .

*Proof of Theorem 10.5.* For the first statement, recall that the square of a Dehn twist along an  $n$ -sphere in a  $(2n)$ -dimensional manifold is smoothly isotopic to the identity. Therefore removing a neighborhood of  $L$ , and gluing it back with  $2m$  Dehn twists – as can be used to describe  $(S^{2n+1}, \xi_{std})_{L, \Phi, 2m}$  – produces a smooth manifold diffeomorphic to  $S^{2n+1}$ , as the gluing map is smoothly isotopic to the identity map on the gluing region.

If the integer  $m$  is less than zero, then by Theorem 9.15(1,2)  $(S^{2n+1}, \xi_{std})_{L, 2m}$  can be obtained from attaching  $-2m$  Weinstein handles to  $(\mathbb{D}^{2n+2}, \lambda_{std})$  which is bounded by  $(S^{2n+1}, \xi_{std})$ . This gives a Stein filling  $(W, \lambda)$  of  $(S^{2n+1}, \xi_{std})_{L, 2m}$  whose Euler characteristic is  $\chi(W) = 1 - 2m \neq 1$ . Therefore,  $W$  cannot be diffeomorphic to a disk and so  $(S^{2n+1}, \xi_{std})_{L, \Phi, 2m}$  cannot be contact-diffeomorphic to  $(S^{2n+1}, \xi_{std})$  by Theorem 10.6.

Now suppose that the integer  $m$  is greater than zero. In this case, we can cancel the contact surgeries used to define  $(S^5, \xi_{std})_{L, \Phi, 2m}$  by a contact  $(-1/m)$ -surgery along a push-off of  $L$ . This surgery can be realized as a sequence of Weinstein handle attachments, yielding a Stein cobordism  $(W, \lambda)$  from  $(S^{2n+1}, \xi_{std})_{L, \Phi, 2m}$  to  $(S^{2n+1}, \xi_{std})$ . If  $(S^{2n+1}, \xi_{std})_{L, \Phi, 2m}$  is equal to  $(S^{2n+1}, \xi_{std})$  then we can fill the concave end of  $(W, \lambda)$  with a standard disk  $(\mathbb{D}^{2n+2}, \lambda_{std})$ . This produces a Stein filling of  $(S^{2n+1}, \xi_{std})$  whose Euler characteristic is  $\chi(W \cup \mathbb{D}^{2n+2}) = 1 + 2m \neq 1$ , contradicting Theorem 10.6.  $\square$

**Remark 10.7.** The same procedure used to construct exotic contact structures on the  $(2n+1)$ -spheres using contact  $(1/2m)$ -surgery along Legendrian spheres in  $(S^{2n+1}, \xi_{std})$  can be used to construct infinite families of potentially distinct contact structures on arbitrary contact manifolds of dimensions 5 and 13.

*Proof of Theorem 1.23.* This is now a simple application of Proposition 3.11 and Theorem 10.5. If  $\tau_n^2 \in \text{Symp}(\mathbb{D}^*S^n, -d\lambda_{can}, \partial\mathbb{D}^*S^n)$  was isotopic to the identity mapping, then the contact manifold  $(S^{2n+1}, \xi_{std})_{L, \Phi, 2m}$  would be contact diffeomorphic to  $(S^{2n+1}, \xi_{std})$  as the convex gluing instructions used to define this contact manifold – by identifying the boundaries of  $(S^{2n+1} \setminus N(L), \xi_{std})$  and  $(N(L), \xi_{std}|_{N(L)})$  – would be trivial by Proposition 3.11.  $\square$

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